Distance-Based Judgment Aggregation of Three-Valued Judgments with Weights

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Abstract

Judgment aggregation theory studies how to amalgamate individual opinions on a set of logically related issues into a set of collective opinions. Aggregation rules proposed in the literature are sparse. All proposed rules consider only two-valued judgments, thus imposing the strong requirement that an agent cannot abstain from giving judgments on any of the issues. All proposed rules are also insensitive to weights that can be assigned to different judgments. We construct a family of weight-sensitive rules for aggregating individual judgment sets with abstentions. We do so by generalizing known distance-based judgment aggregation rules. We study the relations between existing distance-based rules and the rules we propose and the computational complexity of the winner determination problem.

1 Introduction

The theory of judgment aggregation studies the problem of aggregating individual answers to a set of binary interconnected questions, called an *agenda*. The answers, *i.e.*, judgments, given on some of the questions constrain the judgments that can consistently be given to others. Consequently, an agreement on the collective set of answers cannot always be reached by statistical pooling, one-by-one, the individual judgments [List and Polak, 2010].

Judgment aggregation is a relatively new field of social choice and it has been predominantly focused on studying the (im)possibility of rules for aggregation with respect to the fairness rules they can simultaneously satisfy. Few judgment aggregation rules have been constructed: the *premise-based* procedure, proposed in [Kornhauser and Sager, 1993] as "issue-by-issue voting" and studied in [Dietrich and Mongin, 2010; Mongin, 2008], *sequential procedures* [List, 2004; Dietrich and List, 2007; Li, 2010], and *distance-based merging procedures* [Pigozzi, 2006; Miller and Osherson, 2009; Endriss *et al.*, 2010]. All of these aggregation rules are defined for complete sets of judgments, i.e., the agents are not allowed to abstain from judgment. Furthermore, all the proposed rules satisfy the property of *anonymity*. The outcome of an anonymous aggregation rule depends only on the judgment sets being aggregated but not on the identity of the source or the nature of the agenda element. We argue that the proposed rules as such are insufficient to cover all judgment aggregation scenarios.

Consider a team that has to determine whether to purchase a new production robot.[1] The team makes the decision based on several factors such as: is the price affordable, is the robot production capacity adequate, is the robot easy to manipulate, etc. The team consists of a design engineer, a manager of the production unit that will use the robot, a purchasing agent, and a person who will be trained to operate the robot. The agents have different areas of expertise and each can address different domains of the purchasing problem. For instance, the design engineer can justifiably choose not to make a judgment on whether the robot is easy to manipulate, while the purchasing agent and the line manager may have different views about how important the price is, even if they have access to the same information.

In situations like this, not all team members need to give their judgments on all the agenda elements. The expertise of the agents may be distributed over the team members with no one member possessing all the relevant information. Furthermore, even when team members make judgments on the same agenda element, they may weigh their judgments differently. The aggregation of their judgments should account for abstentions, but also for different weights assigned to different judgments. The judgment aggregation rules proposed in the literature are not weight-sensitive and they are not designed to handle abstentions. The aim of this paper is to contribute towards filling this gap.

Frameworks of judgment aggregation in which agents are allowed to abstain from giving some judgments have been proposed in [Gärdenfors, 2006; Dokow and Holzman, 2010], but no aggregation rules were given. The challenge in aggregating three-valued judgments is in the decision on how to treat the case when an agent chooses to make no judgment. The abstentions can be interpreted along two dimensions, that of *semantics* and that of *relation* between abstentions and judgments. Abstaining can mean that the agent does not have enough information to make a judgment at present, that he thinks that a judgment cannot be made on that particular agenda element or maybe that he deems his opinion ir-

[1]This example is taken from [Ilgen *et al.*, 1991]

relevant. The chosen semantics of the abstention determines when a set of judgments that contains abstentions is consistent.

The second dimension of interpretation is the relation between an abstention regarding an agenda element and the judgments on that element. For instance, is the abstention an independent position in addition to "yes" and "no", or is it the half-way position between "yes" and "no"? The relation between abstentions and judgments determines the impact that abstentions have on which collective judgment is selected. There are several possibilities. Consider, for example, seven agents judging an issue p. Four of the agents abstain from making a judgment, two judge "yes" and one judges "no". On one hand, the collective judgment for p should be "yes" because this is the position of the majority of the agent's who do make a judgment. On the other hand, the majority of the agents abstain so the group should also abstain from giving a collective judgment on p.

In addition to rules that handle abstentions, we want to construct weight-sensitive rules. The only trivially weight-sensitive judgment aggregation rule considered in the literature is the *dictatorship rule*. Outside of judgment aggregation, weights associated with an agent have been considered in merging information by [Revesz, 1995], and we take the same approach. However, in addition to agent-associated weights, we also consider weights associated with a judgment, thus assigned to a $(judgment,\ agenda\ element)$ pair.

We develop our rules by generalizing the distance minimization approach to judgment aggregation since this approach is applicable to any agenda.[2] In contrast, the sequential aggregation rules are applicable only when there is a total order over the elements of the agenda, while the premise-based approach is applicable when the agenda can be partitioned to a set of *premises* and a set of *conclusions*. Moreover, as we show, the premise-based approach can be emulated by a distance-based aggregation rule.

This paper is structured as follows. In Section 2 we give the necessary preliminaries. In Section 3 the distance-based rules with abstentions are presented, and in Section 4 we propose the weight-sensitive version of these rules. In Section 5 we discuss the introduced rules and the computational complexity of the winner determination problem. In Section 6 we present our conclusions.

2 Preliminaries

There are two types of judgment aggregation frameworks: logic-based, [Dietrich, 2007], and abstract algebraic, [Rubinstein and Fishburn, 1986; Dokow and Holzman, 2010]. In a logic-based framework, the agenda is a set of formulas from a given logic. The agenda is closed under negation and a judgment set in this framework is a consistent subset of the agenda. In an abstract framework no agenda is given, instead, the agents choose from a set of allowed binary sequences. For example, if the agenda of the aggregation problem, in propositional logic, were $\{p, \neg p, p \rightarrow q, \neg(p \rightarrow q), q, \neg q\}$, then the corresponding set of allowed sequences in an abstract framework would be $\{\langle 0,1,0\rangle, \langle 0,1,1\rangle, \langle 1,0,0\rangle, \langle 1,1,1\rangle\}$. *E.g.,* $\{\neg p, p \rightarrow q, q\}$ is a judgment set for this agenda but $\{p, p \rightarrow q, \neg q\}$ is not.

Abstentions can be represented in several ways depending on the framework used. In a propositional logic framework, one can introduce a new agenda element $\overline{p}$ for each pair $\{p, \neg p\}$ to represent "the agent makes no judgment on p" while imposing the additional consistency constraints to denote that neither $\{\overline{p}, p\}$ nor $\{\overline{p}, \neg p\}$ are consistent sets. With this approach there is no need to extend the existing judgment aggregation rules and one can skip directly to constructing weight-sensitive rules. However, adding agenda elements in this way, as we show in Section 5, taxes the time it takes to compute the collective judgment set. [Dokow and Holzman, 2010] use a special symbol $*$, which is interpreted as a variable taking values from $\{0, 1\}$, to represent abstentions in an abstract aggregation framework. This approach, as is the case with any abstract argumentation framework, requests for all of the allowed judgment sets to be explicitly given. The number of possible judgment sequences is exponential with respect to the cardinality of the sequences considered and taxes the space it takes to compute the collective judgment set.

We choose to use a ternary logic-based framework, in which the consistency of a judgment set is determined by a consequence relation. This allows us to keep the agenda as a set not closed under negation, and removes the need for all of the allowed judgment sets, or sequences, to be explicitly stated and stored.

2.1 Ternary logic framework

The choice of a three-valued logic determines the semantics of the abstention. In the ternary logic of Łukasiewicz, [Łukasiewicz, 1920; Urquhart, 2001], the third value is $\frac{1}{2}$, set in the middle of 0, *i.e.*, "false" and 1, *i.e.*, "true". This third value denotes "to be determined later". The Łukasiewicz semantics corresponds to the semantics of the symbol $*$ used by [Dokow and Holzman, 2010]. In the ternary logic of Kleene, [Kleene, 1938], the values that a formula can take are $\{T, I, F\}$, where the third value I denotes "undefined", for this logic also the numerical value set $\{0, \frac{1}{2}, 1\}$ is used with $I \equiv \frac{1}{2}$. In the context of judgment aggregation, the "to be determined later" means that when an agent is abstaining it is because he does not know the value of the agenda element at the moment of casting judgment, "undefined" means that the abstaining agent does not think that a judgment on the agenda element can be made. Other choices for ternary logics can also be made. For instance, the ternary logic of Bochvar interprets the third value as "meaningless" and any formula that has a meaningless component as meaningless, [Urquhart, 2001].

The choice of semantics can be based on the aggregation context in which the rule is used. For instance, the logic of Łukasiewicz is better suited to dynamic aggregation contexts in which agents give judgments to the same agenda several times, since the agents can make a judgment on p in the second round, even though they abstained in the firs. For the

[2] Note that aggregating multi-valued information by distance based merging has been already considered in the literature [Condotta *et al.*, 2008; Coste-Marquis *et al.*, 2007], but only outside of judgment aggregation.

same reason, the Kleene logic can be considered suited for aggregation problems in which the judgments are made only once.

We give a short overview of the logics of Kleene and Łukasiewicz. The syntax of the both of propositional logic (in BNF) $\mathcal{L}_{Prop}$:

$$\varphi ::= \top \mid \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \varphi \leftrightarrow \varphi,$$

where $p \in \mathcal{L}_0$ ranges over the set of atomic formulas. The formulas of $\mathcal{L}_{Prop}$ are assigned values from the set $T = \{0, \frac{1}{2}, 1\}$; $v(\top) = 1$ and $v(\bot) = 0$. The semantics of the non-atomic formulas according to Łukasiewicz is: $v(\neg\varphi) = 1 - v(\varphi)$; $v(\varphi_1 \wedge \varphi_2) = min(v(\varphi_1), v(\varphi_2))$; $v(\varphi_1 \vee \varphi_2) = max(v(\varphi_1), v(\varphi_2))$; $v(\varphi_1 \rightarrow \varphi_2) = min(1, 1 - v(\varphi_1) + v(\varphi_2))$ and $v(\varphi_1 \leftrightarrow \varphi_2) = 1 - |v(\varphi_1) - v(\varphi_2)|$.

The semantics according to Kleene is: $v(\neg\varphi) = 1 - v(\varphi)$; $v(\varphi_1 \wedge \varphi_2) = min(v(\varphi_1), v(\varphi_2))$; $v(\varphi_1 \vee \varphi_2) = max(v(\varphi_1), v(\varphi_2))$; with $\varphi_1 \rightarrow \varphi_2 \equiv \neg\varphi_1 \vee \varphi_2$ and $\varphi_1 \leftrightarrow \varphi_2 \equiv (\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$.

The consequence operator for a ternary logic, *e.g.*, $\models_L$ and $\models_K$, is defined in the standard way, [Urquhart, 2001]. Given a set of formulas $\Gamma \subset \mathcal{L}_{Prop}$ and a formula $\psi \in \mathcal{L}_{Prop}$, ψ is entailed by Γ, if for all assignments v, if $v(\psi) = 1$ for all formulas $\psi \in \Gamma$, then $v(\varphi) = 1$. A formula ψ for which $\emptyset \models_L \psi$ is a tautology of the Łukasiewicz logic, if $\emptyset \models_K \psi$ then ψ is a tautology of the Kleene logic. If $\Gamma \models_L \bot$, then Γ is inconsistent in the Łukasiewicz logic. If $\Gamma \models_K \bot$ then Γ inconsistent in the Kleene logic. *E.g.,* $p \rightarrow p$ is a tautology of the Łukasiewicz logic, but not of the Kleene logic.

The Łukasiewicz logic, together with $\models_L$, is a member of the set of general logics defined [Dietrich, 2007], thus for this logic all the impossibility results shown by [Dietrich, 2007] hold. The Kleeny logic is not a member of this set of general logics. We give the basic definitions using the Łukasiewicz logic framework. The definitions using any other ternary logic framework can be constructed in the same way.

2.2 Judgment aggregation definitions

A judgment aggregation problem is specified by a sequence of logically related issues called an agenda A. In our framework, the issues are well-formed formulas of $\mathcal{L}_{Prop}$. The logic relations between the agenda issues can also be given in addition to the agenda, by a set of formulas $\mathcal{R}$. For example, in the agenda $\mathcal{A} = \{a_1, a_1 \rightarrow a_2, a_2\}$ the elements are logically related, but in the well-known judgment aggregation problem, the "doctrinal paradox of [Kornhauser and Sager, 1993] where $\mathcal{A} = \{a_1, a_2, a_3\}$, the relations of the elements are specified by the additional set of formulas $\mathcal{R} = \{(a_1 \wedge a_2) \leftrightarrow a_3\}$ is given.

A judgment for issue $a \in \mathcal{A}$ is a valuation $v : \mathcal{L}_{Prop} \mapsto \{0, \frac{1}{2}, 1\}$. Note that by adopting this definition we consider an abstention as a judgment. Given a set of n agents N, the judgments rendered by an agent $i \in N$ on all m issues of $\mathcal{A}$ is called a *judgment sequence* $A_i \subseteq \{0, \frac{1}{2}, 1\}^m$. $A(a_j)$ is the judgment on element a_j according to sequence A. We can always create a judgment set A° from a judgment sequence A, and *vice versa.* We say that a judgment sequence A corresponds to a judgment set A° if and only if, for all issues $a \in \mathcal{A}$: $a \in A^\circ$ if and only if $v(a) = 1$; $\neg a \in A^\circ$ if and only if $v(a) = 0$; $a \notin A^\circ$ and $\neg a \notin A^\circ$ if and only if $v(a) = \frac{1}{2}$.

It is usually assumed, and we assume it here, that the judgment sets of all agents are consistent with respect to $\mathcal{R}$ *i.e.*, $\mathcal{A}_i^\circ \cup \mathcal{R} \not\models_L \bot$. The set of all judgment sets which are consistent with respect to $\mathcal{A}$ and $\mathcal{R}$ are denoted by $\Phi^\circ(\mathcal{A}, \mathcal{R}, \models_L)$; the set of its corresponding judgment sequences is denoted by $\Phi(\mathcal{A}, \mathcal{R}, \models_L)$. When $\mathcal{A}$, $\mathcal{R}$ and $\models_L$ are clear we write simply Φ and Φ°. Any subset of Φ which satisfies constraints X is denoted $\Phi^{\downarrow X}$. *E.g.,* the subset of Φ in which all sequences contain only judgments from $\{0, 1\}$ is denoted by $\Phi^{\downarrow\{0,1\}}$.

A *profile* is a $n \times m$ matrix $\pi = [p_{i,j}]$, $p_{i,j} \in \{0, \frac{1}{2}, 1\}$ containing judgments of all agents $i \in N$ over all agenda issues $a \in \mathcal{A}$. If the profile consists only of consistent judgment sequences, then $\pi \in \Phi^n$. A line in the matrix, denoted π_i, corresponds to agent i's judgment sequence. A column in the matrix, denoted π^j, corresponds to the vector of all judgments rendered for $a_j \in \mathcal{A}$.

A judgment aggregation function, for a set of n agents is a function $f : \Phi^n \mapsto \Phi$. A judgment aggregation rule is a correspondence $F : \Phi^n \mapsto \mathcal{P}(\Phi)$, where $\mathcal{P}(\Phi)$ is the power set of Φ. A judgment sequence that is outputted from an aggregation rule is called a *collective judgment sequence.*

3 Distance-based judgment aggregation

A distance-based judgment aggregation procedure is, according to [Endriss *et al.*, 2010], a judgment aggregation rule $DBP : (\Phi^{\downarrow\{0,1\}})^n \mapsto \mathcal{P}(\Phi^{\downarrow\{0,1\}})$ defined as:
$DBP(\pi) = \arg\min_{A \in \Phi^{\downarrow\{0,1\}}} \sum_{i=1}^n \sum_{j=1}^m |(A(a_j) - p_{i,j}|$.
DBP can be generalized, in the style of the belief distance-based merging operators, see for example [Konieczny *et al.*, 2004], to a judgment aggregation rule $D^{d,\odot}$ by replacing the *aggregation function* $\sum$ with a general aggregation function $\odot$ and the Hamming *distance* by some distance d.

An aggregation function $\odot : (\mathcal{R}^+)^n \mapsto \mathcal{R}^+$ is any function that satisfies non-decreasingness, minimality and identity. The function $\odot$ is non-decreasing when, if $x \leq y$, then $\odot(x_1, \ldots, x, \ldots, x_n) \leq \odot(x_1, \ldots, y, \ldots, x_n)$. It satisfies minimality when $\odot(x_1, \ldots, x_n)$ has a unique absolute minimum $k \geq 0$ for $x_1 = \ldots = x_n = 0$ and identity when $\odot(x, \ldots, x) = x$. A distance $d : \{0,1\}^m \times \{0,1\}^m \mapsto \mathbb{R}^+$ is any total function which, for any $A, A' \in dom(d)$, satisfies: $d(A, A') = 0$ if and only if $A = A'$; $d(A, A') = d(A', A)$ and $d(A, A') + d(A', A'') \geq d(A, A'')$. The most common $\odot$ are $\sum$ and max, while the most common d are the Hamming distance, and the drastic distance d_D. The latter is defined as $d_D(A, A') = 0$ if and only if $A = A'$, and $d_D(A, A') = 1$ otherwise.

It is straightforward to extend the rule $D^{d,\odot}$ to aggregate three-valued judgment sequences.

Definition 1 *The three-valued distance-based judgment aggregation rule* $\Delta^{d,\odot}$ *is a rule* $\Delta^{d,\odot} : \Phi^n \mapsto \mathcal{P}(\Phi)$ *such that:* $\Delta^{d,\odot}(\pi) = \arg\min_{A \in \Phi} \odot(d(A, \pi_1), \ldots, d(A, \pi_n))$. *Where* $\odot$ *is as an aggregation function and* d *is a distance.*

Apart from the $\sum$ and the $\max$, we can also use another well known aggregation function, the product $\prod$ [Grabisch *et al.*, 2009], with minor adjustments. We can define the function $\prod$ as $\prod : \prod_{i=1}^{n}(\epsilon + d(A, A_i))$, where $\epsilon \in \mathbb{R}^+$. We need to add the non-null constant ϵ to each distance to avoid multiplying with zero. Observe that $\prod(x_1, \ldots, x_n)$ has a unique absolute minimum in $k = \epsilon$ for $x_1 = \cdots = x_n = 0$.

We give some examples of distance. The drastic distance d_D can be used defined in the same way as for the case of two-valued judgments. The Hamming distance d_H can be defined as $d_H(A, A') = \sum_{i=1}^{m} \delta_h(A(a_i), A'(a_i))$ where $\delta_h(x_1, x_2) = 0$ iff $x_1 = x_2$; $\delta_h(x_1, x_2) = 1$ otherwise. We can use one more well-known distance metric, the *taxicab distance*[3] d_T. The d_T is defined as $d_T(A, A') = \sum_{i=1}^{m} |A(a_i) - A'(a_i)|$. As it can be observed, the d_T collapses into the d_H whenever both the judgment sequences compared are from $\Phi^{\downarrow\{0,1\}}$.

3.1 Basic judgment aggregation properties of $\Delta^{d,\odot}$

The basic properties considered for judgment aggregation are *universal domain*, *anonymity* and *independence of irrelevant alternatives*(IIA) [List and Polak, 2010]. Universal domain is satisfied when the domain of the aggregation rule includes Φ. (IIA) is satisfied when the collective judgment on any $a_j \in \mathcal{A}$ depends only on π^j. Anonymity is satisfied when the collective judgment set for a profile π is the same as the the collective judgment set of any permutation $\sigma(\pi)$.

The properties of universal domain, anonymity and independence of irrelevant alternatives can be extended to apply to aggregation rules as well. The rule $\Delta^{d,\odot}$ satisfies universal domain by construction. The independence of irrelevant alternatives does not hold for $\Delta^{d,\odot}$ and can be demonstrated by an example.

Whether $\Delta^{d,\odot}$ satisfies anonymity depends only on the selected aggregation function $\odot$ and not on the choice of distance. This is because all distances are by definition symmetric functions. $\Delta^{d,\odot}$ satisfies anonymity if and only if $\odot$ is symmetric. When π is a profile and $\hat{\pi} = \sigma(\pi)$ its permutation, observe that if $\hat{\pi} = \sigma(\pi)$ then $(d(\hat{A}, \hat{\pi}_1), \ldots, d(\hat{A}, \hat{\pi}_n))$ is a σ permutation of $(d(\hat{A}, \pi_1), \ldots, d(\hat{A}, \pi_n))$, because $d(\hat{A}, \pi_i) = d(\hat{A}, \hat{\pi}_j)$ when $\pi_i = \hat{\pi}_j$. Consequently $\Delta^{d,\odot}(\pi) = \Delta^{d,\odot}(\hat{\pi})$ if and only if $\odot(\mathbf{x}) = \odot(\sigma(\mathbf{x}))$, $\mathbf{x} \in (\mathbb{R}^+)^n$. An aggregation function is *symmetric* when for all permutations σ, $\odot(\mathbf{x}) = \odot(\sigma(\mathbf{x}))$ (pg.22, [Grabisch *et al.*, 2009]).

All the aggregation functions we considered: max, $\sum$ and $\prod$ are symmetric. Thus $\Delta^{d,\odot}$ is symmetric for all pairs of $\odot \in \{max, \sum, \prod\}$, $d \in \{d_D, d_H, d_T, m\}$.

3.2 Distances and judgment-abstention relations

The impact of the abstentions on the collective judgments is determined by the selection of the distance d. The distance determines the relation between a judgment sequence with abstentions and one without. By choosing the Hamming or the drastic distance, the abstentions are treated as a third option, an alternative to "yes" and "no". The Taxicab distance treats the abstention as a position half-way between "yes" and "no". All of these distances allow for the possibility of an abstention to be part of the collective judgment set. More "distance" functions can be defined for the abstention to have a different impact. For example, the function m assigns the distance zero from any judgment to the third-value judgment, thus ignoring the abstentions in the profile:
$m(A, A') = \sum_{i=1}^{m} \lfloor |A(a_i) - A'(a_i)| \rfloor$.
The function m is not a distance function, but it can be used to specify a distance-based aggregation rule.

[3] The Taxicab, also known as Manhattan, distance was introduced by Hermann Minkowski (1864-1909).

4 $\Delta^{d,\odot}$ with weights

To be able to specify weight sensitive aggregation rules, we need to introduce a new property for the distance functions, that of *granularity*.

Definition 2 *A distance d is granular, if it can be represented as $d(A, A') = \circledast_{i=1}^{m} \delta(A \nabla a_i, A' \nabla a_i)$, where $\circledast$ is a symmetric aggregation function with a unique minimum in $k = 0$.*

From the distances we considered, d_T and d_H are granular, while d_D is not.

A weight is a number $w_{i,j} \in \mathbb{R}^+$, $w_{i,j} \geq 1$, and it denotes the relevance of the judgment of agent i on $a_j \in \mathcal{A}$. The *weight matrix* $W = [w_{i,j}]_{n \times m}$ is an input to a weight-sensitive distance-based aggregation rules.

The weight can be specified by the agent who makes the judgment or by the agent who aggregates the judgments. Its meaning is determined by the aggregation context. In contexts such as the example for the robot purchase given in the introduction, the weight is specified by the agent who makes the judgment and it denotes the relevance the agent assigns to a particular reason, *i.e.*, issue. An agent can assign a weight to an agenda element to denote his confidence in his judgment.

Weights can be used to encode the reputation an agent has regarding particular agenda elements. In this case the weights are assigned by the agent who aggregates the judgments. We show how weights can be constructed from reputation. Assume that $r_{i,j} \in [0, 1]$ is the normalized reputation of agent i regarding $a_j \in \mathcal{A}$. To construct the weights is to set $w_{i,j} = 1 + r_{i,j}$, thus maintaining that $w_{i,j} \geq 1$. When the reputation of the agent is 0 his weight is 1.

Definition 3 *Let d^g be a granular distance and W a weight matrix. A three-valued distance-based judgment aggregation rule with weights $\Delta_W^{d^g,\odot}$ is a rule $\Delta_W^{d^g,\odot} : \Phi^n \times (\mathbb{R}^+)^{n \times m} \mapsto \mathcal{P}(\Phi)$ such that:*
$\Delta_W^{d^g,\odot}(\pi, W) = \arg\min_{A \in \Phi} \odot_{i=1}^{n} \circledast_{j=1}^{m} w_{i,j} \cdot \delta(A(a_j), p_{i,j})$.

Observe that when an agent has an "untarnished" reputation $r_{i,j} = 1$ for an issue, the weighted aggregation rule would still not treat their judgment as a "veto". To achieve "veto" of one agent on an issue, the weights of the remaining agents on that issue need to be set to zero.

Assuming that we have available only the weight associated to an agent, we can construct a $n \times 1$ *weight vector* $\mathbf{w} = [w_i]$, $w_i \geq 1$. A three-valued distance-based judgment aggregation rule with agent-weights $\Delta_w^{d^g,\odot}$ is then defined as $\Delta_w^{d,\odot}(\pi, \mathbf{w}) = \arg\min_{A \in \Phi} \odot_{i=1}^{n} w_i \cdot d(A, \pi_i)$.

When each agent is an expert on different issues, one may want to consider an agent's judgments only on issues in his area of expertise. The weights can be used to encode *subjective agendas*, *i.e.*, individually designated agenda subset $\mathcal{A}_i$. The weights on an agent i are zero for all agenda issues $a_j \notin \mathcal{A}_i$.

5 Some more properties

We first consider the relations between the distance-based judgment aggregation rules we defined. Let F and F' be two judgment aggregation rules defined over domains $dom(F)$ and $dom(F')$ correspondingly. We say that a F is included in in F', denoted $F \subset F'$, if $dom(F) \cap dom(F') \neq \emptyset$ and for each $\pi \in dom(F) \cap dom(F')$, $F(\pi) \subseteq F'(\pi)$.

Proposition 1 *The following inclusion relations hold* $D^{d,\odot} \subset \Delta^{d,\odot} \subset \Delta_w^{d,\odot}$, $\Delta_w^{d^g,\odot} \subset \Delta_W^{d^g,\odot}$ *and* $\Delta^{d,\odot} \subset \Delta_W^{d,\odot}$.

Proof: $D^{d,\odot} \subset \Delta^{d,\odot}$ holds since $\Phi^{\downarrow\{0,1\}} \subset \Phi$; $\Delta^{d,\odot} \subset \Delta_w^{d,\odot}$ holds since we can use the unary vector $\mathbf{u} = (1, 1, \ldots, 1)$ to achieve $\Delta^{d,\odot}(\pi, \mathbf{u}) = \Delta_w^{d,\odot}(\pi, \mathbf{u})$. $\Delta_w^{d^g,\odot} \subset \Delta_W^{d^g,\odot}$, because we can always represent $\Delta_w^{d,\odot}$ through $\Delta_W^{d^g,\odot}$ by setting $w_{i,j} = v_i$ for all $a_j \in \mathcal{A}$. We can always represent the rule $\Delta^{d,\odot}$ as a $\Delta_W^{d^g,\odot}$ rule by setting $w_{i,j} = 1$. ■

Since $\Delta_W^{d^g,\odot}$ subsumes $\Delta_w^{d^g,\odot}$, $\Delta^{d^g,\odot}$ and $D^{d,\odot}$, we can use it to aggregate the profiles for sets of agents for which different types of weights are available.

5.1 Co-domain restrictions

The co-domain of $\Delta^{d^g,\odot}$, $\Delta_w^{d^g,\odot}$ and $\Delta_W^{d^g,\odot}$ corresponds to the set Φ° of all judgment sets A° for $\mathcal{A}$ for which $A^\circ \cup \mathcal{R}$ is consistent. Consequently, the selected judgment sequences may have the undecided judgments in them, and the sequence in which all judgments are undecided is also a possible outcome. This might be undesirable, and one may want to allow only for sequences from $\{0, 1\}^m$ to be in the co-domain of the aggregation rule.

Ensuring that the aggregate satisfies certain constraints X, such as containing only judgments from $\{0, 1\}$, can be accomplished by restricting co-domain of the rule. The co-domain restricted $\Delta_W^{d^g,\odot}$ can be defined as:

$$\Delta_W^{d^g,\odot}(\pi, W, X) = \arg\min_{A \in \Phi^{\downarrow X}} \odot_{i=1}^{n} \circledast_{j=1}^{m} w_{i,j} \cdot \delta(A(a_j), p_{i,j}).$$

5.2 The premise-based procedure emulated

Restricting the co-domain can be used to engineer certain properties for certain issues, such as for example *adherence to majority*. A judgment $v(a)$ on $a \in \mathcal{A}$ adheres to majority, with respect to a profile π, if the number n_i of agents in π^j, for which $p_{i,j} = v(a_j)$ is greater then the number of agents n_j for which $p_{i,j} \neq v(a_j)$; $v(a) = \frac{1}{2}$ when $n_i = n_j$.

As we know from the impossibility results of [Dietrich, 2007], a judgment set A in which the collective judgment for each issue $a \in \mathcal{A}$ corresponds to the majority judgment in π^j may be such that $A^\circ \cup \mathcal{R} \models_L \perp$. However, for some subset of agenda issues, majority-adherence can be consistently guaranteed. For example, the premise-based procedure guarantees majority-adherence to a subset of the agenda called *premises*. Given a profile π, and a subset of selected issues $b \in \mathcal{A}$, we can define $\Phi^{\downarrow X}$ to be the subset of Φ in which all judgment sequences A are such that which $A(b)$ is majority-adherent with respect to π.

5.3 General complexity result for distance-based judgment aggregation

The *judgment distance-based winner determination problem* for agenda $\mathcal{A}$, set of rules $\mathcal{R}$, and a distance-based rule $\Delta^{d,\odot}$, is defined as follows:

Definition 4 (WinDet for $\Delta^{d,\odot}$)
Input: *Profile* $\pi \in (\Phi(\mathcal{A}, \mathcal{R}, \models_L))^n$, *sequence* $A \in \Phi(\mathcal{A}, \mathcal{R}, \models_L)$.
Output: *true iff* $A \in \Delta^{d,\odot}(\pi)$.

Proposition 2 *If* $\odot$ *and* d *are computable in polynomial time then WinDet for* $\Delta^{d,\odot}$ *is in* Σ_2^P.

We prove the inclusion by showing an algorithm for WinDet.

Algorithm: $\underline{WinDet(\pi, A)}$

1. guess a valuation v for the atoms in $\mathcal{A}$;
2. if v is a model for A and not $ExistBetter(\pi, A)$ then $return(true)$ else $return(false)$;

Oracle: $\underline{ExistBetter(\pi, A)}$

1. guess $A' \in \{0, \frac{1}{2}, 1\}^m$;
2. guess a valuation v' for the atoms in $\mathcal{A}$;
3. if v' is a model for A' and $\odot(d(A', \pi_1), \ldots, d(A', \pi_n)) > \odot(d(A, \pi_1), \ldots, d(A, \pi_n))$ then $return(true)$ else $return(false)$;

Two observations are worth pointing out. In the weighted case, a weight matrix W is also a part of the input. If $\circledast$ and δ are computable in polynomial time wrt the size of π and W, then so is d, and the above result can be easily adapted. Moreover, if the number of possible scores for $\Delta_W^{d,\odot}$ is known in advance and bounded by a polynomial in n, m then computing WinDet for $\Delta_W^{d,\odot}$ is in Θ_2^P (where $\Theta_2^P = \mathbf{P}^{\mathbf{NP}[\log n]}$ is the class of problems solvable by a polynomial-time deterministic Turing machine asking at most $\mathcal{O}(log\ n)$ adaptive queries to an $\mathbf{NP}$ oracle).[4] This can be demonstrated by the following variation of the algorithm. Let Val be the set of possible scores. Also, for an ordered set X, let $med(X)$ denote the median of X, X^+ denote the subset of X from $med(X)$ up, and X^- the part below $med(X)$.

Algorithm: $\underline{WinDet(\pi, A)}$

1. $Poss := Val$;
2. repeat
3. $k := med(Poss)$;
4. if $Exist(\pi, Poss^-)$ then $Poss := Poss^-$ else $Poss := Poss^+$;
5. until $|Poss| = 1$;
6. if $\odot(d(A, \pi_1), \ldots, d(A, \pi_n)) = med(Poss)$ then $return(true)$ else $return(false)$;

Oracle: $\underline{Exist(\pi, Poss)}$

1. guess $A \in \{0, \frac{1}{2}, 1\}^m$ and a valuation v;

[4] We thank an anonymous reviewer for hinting the property and sketching the proof.

2. if v is a model for A and $\odot(d(A,\pi_1),\ldots,d(A,\pi_n)) \in Poss$ then $return(true)$ else $return(false)$;

Again, the algorithm and the result can be easily adapted for the weighted case of $\Delta_W^{d,\odot}$.

6 Conclusions and future work

The literature on judgment aggregation assumes that all agents have to give their judgments on all agenda elements, which seems an important limitation in many scenarios. Moreover, the agents' judgments on the same issue must bear the same weight. In this paper, we make the first step towards filling the gap. Our rules are based on distance minimization, i.e., a rule is specified by an aggregation function $\odot$ and a distance d. Unlike the weight-sensitive distance-based aggregation rules studied in the theory of belief merging by [Revesz, 1995], our weights can be assigned to each pair of $(judgment,\ agenda\ element)$ and not only to agents.

The semantics of abstention is determined by the choice of the propositional ternary logic. Which semantics to choose can be determined by the aggregation setting. The relation of the abstention from judgment to the crisp (yes/no) judgments is determined by the choice of distance d. Formally, we construct a dual judgment aggregation framework based on propositional ternary logic. The framework is dual since we can represent the input from the agents both as subsets of $\overline{\mathcal{A}} = \{\neg a | a \in \mathcal{A}\} \cup \mathcal{A}$, and as a sequence from $\{0, \frac{1}{2}, 1\}^m$. We demonstrate the expressive power of our rules by showing how the co-domain can be constrained to ensure collective judgment sequences with desirable properties.

The worst-case complexity for computing the winner determination problem turns out to be at most Σ_2^P in general, and at most Θ_2^P under reasonable conditions. Note that the specific complexity bounds may depend on the actual choice of d and $\odot$. For example, the WinDet problem for the drastic distance d_D can be solved in linear time with respect to the number of agents and issues. In the future we intend to study further the properties of different $\Delta_W^{d,\odot}$ rules with respect to the choice of $(d,\odot)$.

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