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# Majority-preserving judgment aggregation rules

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**Abstract** The literature on judgment aggregation has now been moving from studying impossibility results regarding aggregation rules towards studying specific judgment aggregation rules. Here we focus on a family of rules that is the natural counterpart of the family of Condorcet-consistent voting rules: *majority-preserving judgment aggregation rules.* A judgment aggregation rule is majority-preserving if whenever issue-wise majority is consistent, the rule should output the majoritarian opinion on each issue. We provide a formal setting for relating judgment aggregation rules to voting rules and propose some basic properties for judgment aggregation rules. We consider six such rules some of which have already appeared in the literature, some others are new. For these rules we consider their relation to known voting rules, to each other, with respect to their discriminating power, and analyse them with respect to the considered properties.

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## 1 Introduction

Judgment aggregation studies the problems related to aggregating a finite set of individual judgments, cast on a collection of logically interrelated issues called an agenda. Judgment aggregation can be seen as a generalisation of preference aggregation and voting. The formal connection between judgment aggregation and preference aggregation makes use of the *preference agenda* [5]: given a set of alternatives $C$, this agenda is composed of propositions of the form "$x$ is preferred to $y$", where $x$ and $y$ are alternatives in $C$; a profile corresponds to a set of individual judgments, whose consistency condition corresponds to the transitivity of the individual votes. This connection raises several natural questions:

1. when can we say that a judgment aggregation rule is a generalisation of a voting rule?
2. how can we lift properties from voting rules to judgment aggregation rules?
3. how can we classify judgment aggregation rules with respect to the properties they satisfy?

Before considering these questions, certain observations regarding the judgment aggregation literature have to be made. This literature has focused considerably more on studying impossibility theorems, rather than on developing and studying specific aggregation rules, a field development approach that departs from the, admittedly much older, field of voting theory. Nevertheless, several recent, independent, works have started to explore the zoo of interesting, concrete judgment aggregation rules, beyond the well known premise-based and conclusion-based rules. Recall that the premise- and conclusion-based rules can only be applied if there exists a prior labelling of the agenda issues as premises and conclusions. The following rules are defined for any agenda: quota-based rules [6], distance-based rules [17,10,8], or rules that are based on the maximisation of some scoring function [19,18,13,4,25].

Some of the newly proposed rules are defined by an obvious analogy with well-known voting rules. *E.g.,* the so-called Young rule [13], that looks for a minimum number of agents to remove so that the resulting profile becomes majority-consistent, is the obvious counterpart of the Young voting rule. For a few other rules, the analogy remains clear, but the formal connection is less trivial to establish. The generalisation of scoring rules to judgment aggregation [4] falls in that category. For a few other rules, the analogy itself is not obvious.

Questions 1 and 2 are highly related and nontrivial. The answer primarily depends on whether the collective judgment set should be consistent with the transitivity constraint, or only with the constraint expressing the existence of a non-dominated alternative. Question 3 can be answered in a similar way as in voting theory. That is, voting rules can be classified according to the property they satisfy (such as Condorcet-consistency) or to their informational needs: for instance, so-called "C1" rules are voting rules for which knowing the pairwise majority relation between alternatives is enough to determine the winner(s); for "C2" rules, the information needed to determine the winner(s) is the number of voters who rank $x$ ahead of $y$, for all pairs of alternatives $x$ and $y$; for scoring rules, the information needed is the number of voters who rank alternative $x$ in position $k$ for all $x$ and $k$; and so on. From the questions we outlined, the first and third question are evidently not such that can be fully addressed in a single paper.

We focus here on proposing the necessary formalisms for addressing these questions and we focus our efforts on *one* property, majority-preservation. Majority-preservation is the natural generalisation to judgment aggregation of the arguably most important property in voting, *Condorcet-consistency.* While a voting rule is Condorcet-consistent if it outputs the Condorcet winner (and no other alternative) whenever there exists one, a judgment aggregation rule is *majority-preserving* if it outputs the majoritarian judgment set (determined by issue-wise majority) whenever it is consistent. In addition, we also generalise two also important properties from voting to judgment aggregation: *unanimity* and *monotonicity.*

We follow earlier work in judgment aggregation [15] in using a constraint-based version of judgment aggregation to represent properties like transitivity of preferences. As it is common in voting theory, we use irresolute rules (also called 'correspondences') rather than functions, that is, a rule outputs a non-empty set of collective judgments. This represents, as common in voting, that there may be a tie between the collective judgments: an irresolute judgment aggregation rule is a function that given an agenda, a constraint, a number of voters, and for each voter an individual judgment (a subset of the agenda consistent with the constraint), gives a set of collective judgments (subsets of the agenda consistent with the constraint). We address the question of relations between voting and judgment aggregation rules in full detail: we define a formal way of mapping a judgment aggregation rule to *two* voting rules, obtained by requiring the collective judgment to be consistent with the transitivity constraint, or with the (weaker) existence of a non-dominated alternative. It is rather intriguing to see which pairs of well-known voting rules correspond to the same judgment aggregation rule. For instance, as we show, the Copeland rule comes together with the Slater rule, whereas the maximin rule comes together with the "ranked pairs" rule.

The last objective of the paper is to compare the majority-preserving judgment aggregation rules along several dimensions and criteria. As it is common in voting theory, we first compare the rules according to their discriminative power: for each pair of rules $F$ and $F'$, we identify whether the set of judgment sets resulting from the application of $F$ is contained in the set of judgment sets resulting from the application of $F'$, or *vice versa*, or if they are incomparable. We later consider whether they satisfy the rest of the unanimity and monotonicity properties.

The outline of the paper is as follows. The general definitions are given in Section 2. In Section 3 we review the rules we study in the paper and show that they are majority-preserving; we then address inclusion and non inclusion relationships between these rules and relate them to the existing literature; we also mention a few other rules which are not majority-preserving (and are thus excluded from our study). In Section 4 we formally relate judgment aggregation rules to voting rules, and identify the voting rules corresponding to each of the judgment rules studied. In Section 5 we study the rules from the point of view of two key properties (unanimity and monotonicity). We discuss further research in Section 6. Related work is discussed throughout the sections, where it best applies.

## 2 Preliminaries

This section introduces the framework we use for judgement aggregation.

Let $\mathcal{L}$ be a set of well formed propositional logic formulas, including $\top$ and $\bot$, built from a set of propositional variables $\mathcal{L}_p$ using the standard connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$. A *pre-agenda* $[\mathcal{A}] \subseteq \mathcal{L}$ is a set of formulas $\{\varphi_1, \ldots, \varphi_m\}$ such that no $\varphi_i \in [\mathcal{A}]$ has the form $\neg\psi$ for some $\psi \in \mathcal{L}$ (that is, no formula in $[\mathcal{A}]$ is a negation of a formula from $\mathcal{L}$). The *agenda* $\mathcal{A} \subseteq \mathcal{L}$ based on a pre-agenda $[\mathcal{A}] \subseteq \mathcal{L}$ is defined as $\mathcal{A} = [\mathcal{A}] \cup \{\neg\varphi \mid \varphi \in [\mathcal{A}]\}$. For $\varphi \in \mathcal{A}$, we define $\overline{\varphi} = \neg\varphi$ if $\varphi \in [\mathcal{A}]$ and $\overline{\varphi} = \psi$ if there exists $\psi \in [\mathcal{A}]$ such that $\varphi = \neg\psi$.

A *constraint* $\Gamma \in \mathcal{L}$ is a formula of $\mathcal{L}$. We say that a set $S$ is $\Gamma$*-consistent* if and only if $S \cup \{\Gamma\}$ is consistent. A set $J \subseteq \mathcal{A}$ is said to be *complete* if and only if for every $\varphi \in [\mathcal{A}]$ we have $\varphi \in J$ or $\neg\varphi \in J$. $J \subseteq \mathcal{A}$ is incomplete if and only if it is not complete. $J \subseteq \mathcal{A}$ is a *judgment set based on* $\mathcal{A}$ *and* $\Gamma$, for short a $(\mathcal{A}, \Gamma)$*-judgment set*, if and only if it is $\Gamma$-consistent and complete. We denote by $\mathcal{D}(\mathcal{A}, \Gamma)$ the set of all $(\mathcal{A}, \Gamma)$-judgment sets.

An $n$-voter *profile* based on $\mathcal{A}$ and $\Gamma$ is a collection of $n$ $(\mathcal{A}, \Gamma)$-judgment sets $P = \langle J_1, \ldots, J_n\rangle$, that is, $J_i \in \mathcal{D}^n(\mathcal{A}, \Gamma)$. $Q$ is a *sub-profile* of $P = \langle J_1, \ldots, J_n\rangle$ if $Q = \langle J_i \mid i \in I\rangle$ for some nonempty subset $I$ of $\{1, \ldots, n\}$.

For two complete judgment sets $A$ and $B$ over the same agenda, the *Hamming distance between* $A$ *and* $B$ is defined as $d_H(A, B) = |A \setminus B|$ $(= |B \setminus A|)$.

For $I \subseteq \mathcal{A}$, we define $\texttt{Comp}_{\mathcal{A},\Gamma}(I)$ as the set of all $(\mathcal{A}, \Gamma)$-judgment sets containing $I$, i.e. $\texttt{Comp}_{\mathcal{A},\Gamma}(I) = \{J \in \mathcal{D}(\mathcal{A}, \Gamma) \mid I \subseteq J\}$. For $S = \{I_1, \ldots, I_k\}$ with $I_1 \subseteq \mathcal{A}, \ldots, I_k \subseteq \mathcal{A}$, we define $\texttt{Comp}_{\mathcal{A},\Gamma}(S) = \cup_{I \in S}\texttt{Comp}_{\mathcal{A},\Gamma}(I)$.

$N(P, \varphi)$ is the number of agents in $P$ with judgment sets that contain $\varphi$, i.e. $N(P, \varphi) = |\{i \mid J_i \in P, \varphi \in J_i\}|$.

*Example 1* Consider the preagenda $[\mathcal{A}] = \{p \wedge r, q, p \wedge q\}$. The corresponding agenda is $\mathcal{A} = \{p \wedge r, \neg(p \wedge r), q, \neg q, p \wedge q, \neg(p \wedge q)\}$. Let $\Gamma = \{q \rightarrow r\}$ be a constraint for $\mathcal{A}$. The set $\mathcal{D}(\mathcal{A}, \Gamma)$ is

$$\mathcal{D}(\mathcal{A}, \Gamma) = \left\{ \begin{array}{ll} \{\neg(p \wedge r), \neg q, \neg(p \wedge q)\}, & \{\neg(p \wedge r), q, \neg(p \wedge q)\}, \\ \{(p \wedge r), \neg q, \neg(p \wedge q)\}, & \{p \wedge r, q, p \wedge q\} \end{array} \right\}$$

An example of $P \in \mathcal{D}^n(\mathcal{A}, \Gamma)$, where $n = 3$ is $P = \langle\{\neg(p\wedge r), q, \neg(p\wedge q)\}, \{\neg(p\wedge r), q, \neg(p \wedge q)\}, \{(p \wedge r), \neg q, \neg(p \wedge q)\}, \{p \wedge r, q, p \wedge q\}\rangle$. $Q = \langle\{(p \wedge r), \neg q, \neg(p \wedge$

$q)\}, \{p \wedge r, q, p \wedge q\}\rangle$ is a subprofile of $P$. If $S = \{\{\neg(p \wedge r), \neg(p \wedge q)\}, \{p \wedge r, q\}\}$ then $\texttt{Comp}_{\mathcal{A},\Gamma}(S) = \{\{\neg(p \wedge r), q, \neg(p \wedge q)\}, \{\neg(p \wedge r), \neg q, \neg(p \wedge q)\}, \{p \wedge r, q, p \wedge q\}\}$. Finally, $N(P, p \wedge q) = 1$ and $N(P, \neg(p \wedge q)) = 3$.

Most often we will write profiles in a table, as the one given in Table 1, equivalent to the $P$ we just stated, with the preagenda elements given in the topmost row and the name of the judgment sets in the leftmost column. If a judgment set contains a $\varphi \in [\mathcal{A}]$, then we mark this with a "+" in the table, while if a judgment set contains a $\neg\varphi$ for a $\varphi \in [\mathcal{A}]$, we mark this with a "-" in the table. The constraint, unless $\top$, will be the denoted in the table caption.

We now define what we consider to be a judgment aggregation rule. We give the definition for a variable agenda and constraint, noting that judgement aggregation rules can also be defined for a specific agenda, constraint, but also number of voters.

**Definition 1 (Judgment aggregation rule)** An (irresolute) judgment aggregation rule, denoted by $F$, is a is a function that given a number of voters $n$ and a $n$-voter judgment aggregation profile $P = \langle J_1, \ldots, J_n \rangle$, outputs a nonempty set of judgment sets based on $\mathcal{A}$ and $\Gamma$.

Like in voting theory, resolute rules can be defined from irresolute ones by composing them with a tie-breaking mechanism. We do not pursue this direction here and consider irresolute rules throughout the paper.

We classify the judgment aggregation procedures by distinguishing between rules based on the majoritarian judgment set, rules based on the weighted majoritarian judgment set, and rules based on the removal or change of individual judgments. We give the necessary definitions for such classification.

**Definition 2** The *majoritarian judgment set* associated with profile $P = \langle J_1, \ldots, J_n \rangle$ contains all elements of the agenda that are supported by a strict majority of judgment sets in $P$, *i.e.*,

$$m(P) = \left\{ \varphi \in \mathcal{A} \mid N(P, \varphi) > \frac{n}{2} \right\}$$

where $N(P, \varphi)$ is the number of agents in $P$ with judgment sets that contain $\varphi$, i.e. $N(P, \varphi) = |\{i \mid J_i \in P, \varphi \in J_i\}|$. A profile $P$ based on $\mathcal{A}$ and $\Gamma$ is *majority-consistent* iff $m(P)$ is $\Gamma$-consistent[1].

Roughly, a judgment aggregation rule $F$ is majority-preserving iff $F$ returns only the majoritarian judgment set whenever it is consistent. However, there is one subtlety in the case of ties. For example, when we have agenda $\mathcal{A} = \{p, \neg p, q, \neg q\}$ and individual judgments $J_1 = \{p, q\}$ and $J_2 = \{p, \neg q\}$, then $m(\langle J_1, J_2 \rangle) = \{p\}$, which is shorthand for two collective judgments, namely $\{p, \neg q\}$ and $\{p, q\}$.

[1] The same notion is called *Condorcet consistency* in [19].

*Example 2* Consider again the profile from Example 1. We represent this profile in Table 1. $m(P) = \{q, \neg(p \wedge q)\}$ is an incomplete subset of $\mathcal{A}$, and we have $\texttt{Comp}_{\mathcal{A},\Gamma}(m(P)) = \{\{p \wedge r, q, \neg(p \wedge q)\}, \{\neg(p \wedge r), q, \neg(p \wedge q)\}\}$.

**Definition 3 (Majority-preserving)** A judgment aggregation rule $F$ is *majority-preserving* iff for every agenda $\mathcal{A}$, for every $\Gamma \in \mathcal{L}$, for every majority-consistent profile $P$ based on $\mathcal{A}$ and $\Gamma$, we have $F(P) = \texttt{Comp}_{\mathcal{A},\Gamma}(m(P))$.

| Voters | $p \wedge r$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| $J_1 \times 2$ | - | + | - |
| $J_2 \times 1$ | + | - | - |
| $J_3 \times 1$ | + | + | + |
| $m(P)$ | | + | - |

Table 1: $\Gamma = \{q \rightarrow r\}$.

We use the following running example to illustrate our judgment aggregation rules in this paper.

*Example 3* Consider the pre-agenda $[\mathcal{A}] = \{p \wedge r, p \wedge s, q, p \wedge q, t\}$ and a 17-voter profile $P$ of Table 2. As $m(P) = \{p \wedge r, p \wedge s, q, \neg(p \wedge q), t\}$ is an inconsistent judgment set, $P$ is not majority-consistent.

| Voters | { $p \wedge r$, | $p \wedge s$, | $q$, | $p \wedge q$, | $t$ } |
|---|---|---|---|---|---|
| $J_1 \times 6$ | + | + | + | + | + |
| $J_2 \times 4$ | + | + | - | - | + |
| $J_3 \times 7$ | - | - | + | - | - |
| m(P) | + | + | + | - | + |

Table 2: $\Gamma = \top$

## 3 Judgment Aggregation Rules

We now define three families of judgment aggregation rules: rules based on the majoritarian judgment set, rules based on the weighted majoritarian judgment set and rules based on the removal or change of individual judgments.

### 3.1 Rules based on the majoritarian judgment set

We begin by the family of rules based on the majoritarian judgment set. This family can be viewed as the judgment aggregation counterpart of voting rules that are based on the pairwise majority graph, also known as *C1 rules* in Fishburn's classification [1]. Being based on the majoritarian judgment set means that for any $P$ and $Q$ such that $m(P) = m(Q)$ we have $F(P) = F(Q)$. Now we define several rules based on the majoritarian judgment set.

**Definition 4** Given a set of formulae $\Sigma \subseteq \mathcal{L}$ and formula $\Gamma \in \mathcal{L}$, $S \subseteq \Sigma$ is a maximal $\Gamma$-consistent subset of $\Sigma$ iff $S$ is $\Gamma$-consistent and there exists no $\Gamma$-consistent set $S'$ such that $S \subset S' \subseteq \Sigma$.

A set $S \subseteq \Sigma$ is a maxcard (for "maximal cardinality") $\Gamma$-consistent subset of $\Sigma$ iff $S$ is $\Gamma$-consistent and there exists no $\Gamma$-consistent set $S' \subseteq \Sigma$ such that $|S| < |S'|$.

$max(S, \Gamma, \subseteq)$ denotes the set of all maximal $\Gamma$-consistent subsets of $S$. $max(S, \Gamma, |.|)$ denotes the set of all maxcard $\Gamma$-consistent subsets of $S$.

**Definition 5 (Maximal and maxcard sub-agenda rules)** The maximal sub-agenda (MSA) and the maxcard sub-agenda (MCSA) rules are defined as follows: for every agenda $\mathcal{A}$, for every $\Gamma \in \mathcal{L}$, for every $(\mathcal{A}, \Gamma)$-profile $P$,

$$\text{MSA}_{\mathcal{A},\Gamma}(P) = \texttt{Comp}_{\mathcal{A},\Gamma}(max(m(P), \Gamma, \subseteq)), \tag{1}$$

$$\text{MCSA}_{\mathcal{A},\Gamma}(P) = \texttt{Comp}_{\mathcal{A},\Gamma}(max(m(P), \Gamma, |.|)). \tag{2}$$

Intuitively, the MSA rule operates by removing a minimal set of judgments from $m(P)$, such that a consistent set is obtained. Note that for each $\varphi \in m(P)$, there exists at least one $J \in \text{MSA}(P)$ such that $\varphi \in J$.

*Example 4* Consider the same agenda $\mathcal{A}$ and profile $P$ as in Example 3. The maximal $\Gamma$-consistent subsets of $m(P)$ are $\{p \wedge r, p \wedge s, q, t\}$, $\{p \wedge r, \neg(p \wedge q), t\}$ and $\{q, \neg(p \wedge q), t\}$; therefore

$$\text{MSA}_{\mathcal{A},\top}(P) = \left\{ \begin{array}{rrrr} \{p \wedge r, & p \wedge s, & q, & p \wedge q, t\}, \\ \{p \wedge r, & p \wedge s, & \neg q & \neg(p \wedge q), t\}, \\ \{\neg(p \wedge r), & \neg(p \wedge s), & q, & \neg(p \wedge q), \ t\} \end{array} \right\}.$$

Intuitively, the MCSA rule operates by removing a minimal number of judgments (with respect to cardinality) from $m(P)$ so that a consistent set is obtained. Clearly, all the sets selected by MCSA will also be selected by MSA, but the reverse does not hold, as it can be witnessed from Example 5.

MSA and MCSA are clearly majority-preserving.

*Example 5* Consider again the agenda and profile from Example 3. We obtain

$$\text{MCSA}_{\mathcal{A},\top}(P) = \left\{ \begin{array}{l} \{p \wedge r, p \wedge s, \quad q, \qquad p \wedge q, \ t\}, \\ \{p \wedge r, p \wedge s, \neg q, \neg(p \wedge q), \ t\} \end{array} \right\}.$$

The rule MSA is called "Condorcet admissible set" by Nehring et al. [19][2]. The rule MCSA coincides with the "Slater rule" [19], and with the $\texttt{Endpoint}_{d_H}$

[2] There it is defined in a different way, using quantitative intuitions: given two $(\mathcal{A}, \Gamma)$-judgment sets $J$ and $J'$ and a profile $P$, and letting $J \cap \{\varphi, \neg\varphi\} = \{\varphi_J\}$ and $J' \cap \{\varphi, \neg\varphi\} = \{\varphi_{J'}\}$, they say that $J$ is more representative than judgment set $J'$ of $P$ (denoted $J \succeq_P J'$) if for any $\varphi \in [\mathcal{A}]$, it holds that $N(\varphi_J, P) \geq N(\varphi_{J'}, P)$ — in other terms, $J$ receives at least as much support as $J'$ on every issue. Then they define the Condorcet admissible set of profile $P$ as the set of all $(\mathcal{A}, \Gamma)$-judgment sets that are maximal with respect to $\succeq_P$.

rule [17]. The connection between MCSA and $\texttt{Endpoint}_{d_H}$ is not evident, therefore we repeat the definition of $\texttt{Endpoint}_{d_H}$ using our terminology[3]: for every agenda $\mathcal{A}$, for every $\Gamma \in \mathcal{L}$, for every $(\mathcal{A}, \Gamma)$-profile $P$,

$$\texttt{Endpoint}_{d_H,\mathcal{A},\Gamma}(P) = \mathrm{argmin}_{J \in \mathcal{D}(\mathcal{A},\Gamma)} d_H(J, m(P))$$

It is clear that MCSA $= \texttt{Endpoint}_{d_H}$.

### 3.2 Rules based on the weighted majoritarian judgment set

This family can be viewed as the judgment aggregation counterpart of voting rules that are based on the weighted pairwise majority graph, also known as *C2 rules* in Fishburn's classification [1]. Rules of this family are sensitive to the *number* of agents who support a proposition, whereas rules based on the majoritarian judgment set did not distinguish between close and strong majorities. Formally, $R$ is based on the weighted majoritarian judgment set if for any two profiles $P$ and $Q$ such that $N(P, \varphi) = N(Q, \varphi)$ for all $\varphi \in \mathcal{A}$, we have $R(P) = R(Q)$. Since $m(P)$ can be recovered from $N(P, .)$, any rule based on the majoritarian judgment set is also based on the weighted majoritarian judgment set.

The first rule of this class we consider is the *maxweight sub-agenda rule.*

**Definition 6 (Maxweight sub-agenda rule)** The maxweight sub-agenda rule (MWA) is defined as follows: for every agenda $\mathcal{A}$, for every $\Gamma \in \mathcal{L}$, for every $(\mathcal{A}, \Gamma)$-profile $P$,

$$\mathrm{MWA}_{\mathcal{A},\Gamma}(P) = \underset{J \in \mathcal{D}(\mathcal{A},\Gamma)}{\mathrm{argmax}}\, W_P(J) \quad \text{where} \quad W_P(J) = \sum_{\varphi \in J} N(P, \varphi).$$

The MWA rule appears in many places under different names: "Prototype" [17], "median rule" [19] and "simple scoring rule" [4]. It also appears under a different, but equivalent formulation, under the name "distance-based procedure" [10, 17]. Variants of this rule were defined by Pigozzi [21] and before that by Konieczny and Pino-Pérez [12]. For completeness we give here this equivalent distance-based formulation. Given the Hamming distance $d_H$ between two judgment sets, the distance-based rule $F^{d_H,\Sigma}$ is defined as follows: for every agenda $\mathcal{A}$, for every $\Gamma \in \mathcal{L}$, for every $(\mathcal{A}, \Gamma)$-profile $P$,

$$F^{d_H,\Sigma}_{\mathcal{A},\Gamma}(P) = \underset{J \in \mathcal{D}(\mathcal{A},\Gamma)}{\mathrm{argmin}} \sum_{J_i \in P} d_H(J_i, J).$$

We show that $F^{d_H,\Sigma}$ coincides with MWA. An independent proof that these two rules are equal can be found in the paper by Dietrich [4, Proposition 1].

**Proposition 1** $F^{d_H,\Sigma} \equiv$ MWA.

[3] The original definition of $\texttt{Endpoint}_d$ [17] was more general: it was introduced for an arbitrary distance $d$ between judgment sets.

*Proof* Let $A$ be an agenda, $\Gamma \in \mathcal{L}$, $J, J' \in \mathcal{D}(\mathcal{A}, \Gamma)$ and $\varphi \in \mathcal{A}$. Define $h(\varphi, J, J') = 0$ iff $\varphi \in J \cap J'$, and $h(\varphi, J, J') = 1$ otherwise.

For every $n$-voter $(\mathcal{A}, \Gamma)$-profile, for every $J \in \mathcal{D}(\mathcal{A}, \Gamma)$, we have

$$\sum_{J_i \in P} d_H(J, J_i) = \sum_{J_i \in P} \sum_{\varphi \in J} h(\varphi, J, J_i) = \sum_{\varphi \in J} \sum_{J_i \in P} h(\varphi, J, J_i) = \sum_{\varphi \in J} (n - N(P, \varphi))$$
$$= |J| \cdot n - \sum_{\varphi \in J} N(P, \varphi) = |J| \cdot n - W_P(J).$$

For every $J \in \mathcal{D}(\mathcal{A}, \Gamma)$, it holds that:

1. $J \in F^{d_H, \Sigma}_{\mathcal{A}, \Gamma}(P)$ if and only if $J$ minimises $\sum_{J_i \in P} d_H(J, J_i)$
2. $J \in \text{MWA}_{\mathcal{A}, \Gamma}(P)$ if and only if $J$ maximises $W_P(J)$; i.e. if and only if $J$ minimises $|J| \cdot n - W_P(J)$

Since $\sum_{J_i \in P} d_H(J, J_i) = |J| \cdot n - W_P(J)$, we conclude that $J \in F^{d_H, \Sigma}_{\mathcal{A}, \Gamma}(P)$ if and only if $J \in \text{MWA}_{\mathcal{A}, \Gamma}(P)$.

*Example 6* Consider the agenda and profile of Example 3. We obtain:

| | |
|---|---|
| $N(P, p \wedge r) = 10,$ | $N(P, \neg(p \wedge r)) = 7$ |
| $N(P, p \wedge s) = 10,$ | $N(P, \neg(p \wedge s)) = 7$ |
| $N(P, q) = 13,$ | $N(P, \neg q) = 4$ |
| $N(P, p \wedge q) = 6,$ | $N(P, \neg(p \wedge q)) = 11$ |
| $N(P, t) = 10,$ | $N(P, \neg t) = 7$ |

$\text{MWA}_{\mathcal{A}, \top}(P) = \{\{p \wedge r, p \wedge s, q, p \wedge q, t\}\}$, due to the fact that $W_P(\{p \wedge r, p \wedge s, q, p \wedge q, t\}) = 49$ is maximal with respect to all $J \in \mathcal{D}(\mathcal{A}, \Gamma)$.

**Proposition 2** MWA *is majority-preserving.*

*Proof* Let $\mathcal{A}$ be an agenda, $\Gamma \in \mathcal{L}$ and $P$ an $n$-voter majority-consistent $(\mathcal{A}, \Gamma)$-profile, We claim that $\text{MWA}_{\mathcal{A}, \Gamma}(P) = \texttt{Comp}_{\mathcal{A}, \Gamma}(m(P))$. Note that $J \in \text{MWA}_{\mathcal{A}, \Gamma}(P)$ if and only if $J$ is a judgment set maximising $\sum_{\varphi \in J} N(P, \varphi)$. Let $B = \{\varphi \in [\mathcal{A}] \mid \varphi \notin m(P) \text{ and } \neg\varphi \notin m(P)\}$ and $B' = B \cup \{\neg\varphi \mid \varphi \in B\}$. For every $\varphi \in B'$, we have $N(P, \varphi) = \frac{n}{2}$. Thus, whether $\varphi$ or $\overline{\varphi}$ is in $J$ is irrelevant for the score $\sum_{\varphi \in J} N(P, \varphi)$. On the contrary, for every $\varphi \in \mathcal{A} \setminus B'$, in order to maximise $\sum_{\varphi \in J} N(P, \varphi)$, $J$ must contain $\varphi$ if and only if $\varphi \in m(P)$. Hence, $J \in \text{MWA}_{\mathcal{A}, \Gamma}(P)$ if and only if $J \in \texttt{Comp}_{\mathcal{A}, \Gamma}(m(P))$.

The following rule is inspired from the *ranked pairs rules* in voting theory [23]. It consists in first fixing the truth value for the elements of the agenda with the largest majority. It proceeds by considering the elements $\varphi$ of the agenda in non-increasing order of $N(P, \varphi)$ and fixing each agenda issue value to the majoritarian value if it does not lead to an inconsistency.

**Definition 7 (Ranked agenda)** Let $\mathcal{A} = \{\psi, \ldots, \psi_{2m}\}$ be an agenda, $\Gamma \in \mathcal{L}$ and let $P$ be a $(\mathcal{A}, \Gamma)$-profile. Let $\succsim_P$ be the weak order on $\mathcal{A}$ defined by: for all $\psi, \psi' \in \mathcal{A}$, $\psi \succsim_P \psi'$ iff $N(P, \psi) \geq N(P, \psi')$. For a permutation $\sigma$ of $\{1, \ldots, 2m\}$, let $>_\sigma$ be the linear order on $\mathcal{A}$ defined by $\psi_{\sigma(1)} >_\sigma \cdots >_\sigma \psi_{\sigma(2m)}$. We say that $>_\sigma$ is compatible with $\succsim_P$ if $\psi_{\sigma(1)} \succsim_P \cdots \succsim_P \psi_{\sigma(2m)}$. The ranked agenda rule is defined as follows: $J \in \mathrm{RA}_{\mathcal{A},\Gamma}(P)$ if and only if there exists a permutation $\sigma$ such that $>_\sigma$ is compatible with $\succsim_P$ and such that $J = J_\sigma$ is obtained by the following procedure:

$S := \emptyset$;
for $j = 1, \ldots, 2m$ do
    if $S \cup \{\psi_{\sigma(j)}\}$ is consistent, then $S := S \cup \{\psi_{\sigma(j)}\}$
end for;
$J_\sigma := S$.

Note that RA is based on the weighted majoritarian judgment set.

*Example 7* Consider the profile of Example 3. We have $q \succ_P \neg(p \wedge q) \succ_P p \wedge r \sim_P p \wedge s \sim_P t \succ_P \neg(p \wedge r) \sim_P \neg(p \wedge s) \sim_P \neg t \succ_P p \wedge q \succ_P \neg q$ (where $\sim_P$ and $\succ_P$ are respectively the indifference and the strict preference relations induced from $\succsim_P$). We obtain

$$\mathrm{RA}_{\mathcal{A},\top}(P) = \{\{q, \neg(p \wedge q), t, \neg(p \wedge r), \neg(p \wedge s)\}\}.$$

Note that RA is well-defined in the sense that it outputs a set of (complete) judgment sets.

**Proposition 3** RA *is majority-preserving.*

*Proof* Let $P$ be a majority-consistent profile based on $\mathcal{A}$ and $\Gamma$. We first show that $\mathrm{RA}_{\mathcal{A},\Gamma}(P) \subseteq \mathtt{Comp}_{\mathcal{A},\Gamma}(P)$. Let $>_\sigma$ be a linear order on $\mathcal{A}$ induced by permutation $\sigma$ and compatible with $\succsim_P$. Observe that in $>_\sigma$, the elements of $m(P)$ are considered before the elements of $\mathcal{A} \setminus m(P)$. Therefore, when an element $\varphi$ of $m(P)$ is considered, the current judgment set $S$ is a subset of $m(P)$ and $S \cup \{\varphi\} \subseteq m(P)$, therefore $S \cup \{\varphi\}$ is consistent, which implies that $\varphi$ is incorporated into $S$. Since this is true for every $\varphi \in \mathcal{A}$, we get that every element of $\mathrm{RA}_{\mathcal{A},\Gamma}(P)$ contains $m(P)$.

Let us now show that $\mathtt{Comp}_{\mathcal{A},\Gamma}(P) \subseteq \mathrm{RA}_{\mathcal{A},\Gamma}(P)$. Let $J \in \mathtt{Comp}_{\mathcal{A},\Gamma}(m(P))$. Take $>_\sigma$ such that all elements of $m(P)$ are considered first, then all elements of $J \setminus m(P)$, and then all elements of $\mathcal{A} \setminus J$. This order is compatible with $\succsim_P$, because if $\varphi \in m(P)$ then $N(P, \varphi) > \frac{n}{2}$, if $\varphi \in J \setminus m(P)$ then $N(P, \varphi) = \frac{n}{2}$ and if $\varphi \in \mathcal{A} \setminus J$ then $N(P, \varphi) \leq \frac{n}{2}$. Lastly, $J_\sigma = J$, which proves that $J \in \mathrm{RA}_{\mathcal{A},\Gamma}(P)$.

The RA rule is new, but presents some similarity of the *leximax* rule in [18], which is in fact is a refinement of RA[4] and which has been studied independently in [11].

### 3.3 Rules based on the removal or change of individual judgments

The last family of rules we consider contains rules that are constructed around the principle of minimally changing the aggregated profile. The difference between the original and changed profile is expressed in terms of some distance. Different rules will be obtained with different distance functions. This family of judgment aggregation rules can be viewed as the judgment aggregation counterpart of voting rules that are based on performing minimal operations on profiles with the purpose of obtaining a profile for which a Condorcet winner exists. In [9] these rules are said to be rationalizable by some distance with respect to the Condorcet consensus class.)

The first rule we consider is called the *Young* rule for judgment aggregation, by analogy with the Young rule in voting, which outputs the candidate $x$ minimising the number of voters to remove from the profile so that $x$ becomes a Condorcet winner.

**Definition 8 (Young rule)** Let $\mathcal{A}$ be an agenda, $\Gamma \in \mathcal{L}$ and $P$ an $n$-voter $\mathcal{A}, \Gamma$-profile. We define set $MSP(P)$ as follows: $P_I \in MSP$ if and only if

1. $P_I$ is a $k$-voter sub-profile of $P$
2. $P_I$ is majority-consistent
3. there exists no $P_{I'}$ such that $P_{I'}$ is a $j$-voter majority-consistent sub-profile of $P$ and $j > k$

The Young judgment aggregation rule is defined as

$$\textsc{y}_{\mathcal{A},\Gamma}(P) = \texttt{Comp}(\{m(P_I) \mid P_I \in MSP(P)\}).$$

Intuitively, this rule consists of removing a minimal number of agents so that the profile becomes majority-consistent. Or, equivalently, we maximize the number of voters we keep of a given profile. If the profile $P$ is majority-consistent, then no voter needs to be removed and $\textsc{y}_{\mathcal{A},\Gamma}(P) = \texttt{Comp}_{\mathcal{A},\Gamma}(m(P))$, hence Y is majority-preserving.

[4] Here is a profile $P$ for which RA and *leximax* differ:

| | $p \wedge q$ | $p$ | $q$ | $p \wedge r$ | $q \wedge r$ | $s$ |
|---|---|---|---|---|---|---|
| 5× | − | + | − | + | − | + |
| 5× | − | − | + | − | + | − |
| 4× | + | + | + | + | + | + |
| 1× | + | + | + | − | − | − |

$\textsc{ra}(P) = \{\{p \wedge q, p, q, p \wedge r, q \wedge r, s\}, \{\neg(p \wedge q), p, \neg q, p \wedge r, \neg(q \wedge r), s\}, \{\neg(p \wedge q), \neg p, q, \neg(p \wedge r), q \wedge r, s\}\}$ and $\text{leximax}(P) = \{\{p \wedge q, p, q, p \wedge r, q \wedge r, s\}\}$.

*Example 8* Once again we consider $\mathcal{A}$ and $P$ from Example 3. We obtain $\text{Y}_{\mathcal{A},\top}(P) = \texttt{Comp}_{\mathcal{A},\Gamma}(\{q, \neg(p \wedge q)\}) =$

$$\left\{ \begin{array}{l} \{\neg(p \wedge r),\ \neg(p \wedge s),\ q,\ \neg(p \wedge q),\ \ \ t\}, \\ \{\neg(p \wedge r),\ \neg(p \wedge s),\ q,\ \neg(p \wedge q),\ \neg t\} \end{array} \right\}.$$

This result is obtained by removing 3 copies of $J_1$ or (2 copies of $J_1$ and one copy of $J_2$) or (one copy of $J_1$ and 2 copies of $J_2$) or 3 copies of $J_2$. Removing less judgment sets, or other 3 judgment sets, does not lead to a majority-consistent profile.

The last rule we define does not remove agenda elements and/or voters, but looks for a minimal number of *atomic changes* in the profile so that $P$ becomes majority-consistent. We consider an atomic change to be the change of truth value of one element of the agenda in an individual judgment set. For instance, if $J_1 = \{p, q, p \wedge q, r, p \wedge r\}$, then $J'_1 = \{\neg p,\ q,\ \neg(p \wedge q), r, \neg(p \wedge r)\}$ is obtained from $J_1$ by a series of three atomic changes (change in the truth value of $p$, of $p \wedge q$ and of $p \wedge r$).

Replacing *having a Condorcet winner* by *being majority-consistent* and adapting the notion of elementary change, we get our judgment aggregation rule that corresponds to the $\text{FULL}_d$ rule by Miller and Osherson [17] for the choice of the Hamming distance.

**Definition 9 (Minimal atomic change rule)** Let $\mathcal{A}$ be an agenda, $\Gamma \in \mathcal{L}$ and let $P = \{J_1, ..., J_n\}$ and $Q = \{J'_1, ..., J'_n\}$ be two $n$-voter $(\mathcal{A}, \Gamma)$-profiles. We define:

$$D_H(P, Q) = \sum_{i=1}^{n} d_H(J_i, J'_i).$$

The minimal atomic change rule is defined as:

$$\text{MNAC}_{\mathcal{A},\Gamma}(P) = \texttt{Comp}_{\mathcal{A},\Gamma}(\{m(Q) \mid Q \in \underset{Q' \in \mathcal{D}^n(\mathcal{A},\Gamma)}{\text{argmin}} D_H(P, Q')\}).$$

*Example 9* Consider the same agenda $\mathcal{A}$ and profile $P$ from Example 3 and let $\Gamma = \top$. Profile $Q$ given in Table 3 is the closest majority-consistent profile to $P$ with $D_H(P, Q) = 3$. We obtain $\text{MNAC}_{\mathcal{A},\Gamma_{\mathcal{A},\top}}(P) = \{\{p \wedge r, p \wedge s, q, p \wedge q, t\}\}$.

| Voters | $\{p \wedge r$ | $p \wedge s$ | $q$ | $p \wedge q$ | $t\}$ |
|---|---|---|---|---|---|
| 6× | + | + | + | + | + |
| 4× | + | + | - | - | + |
| 3× | - | - | + | + | - |
| 4× | - | - | + | - | - |
| $m(Q)$ | + | + | + | + | + |

Table 3

If $P$ is majority-consistent then no elementary change is needed, therefore MNAC is majority-preserving.

We now establish the (non)inclusion relationships between the rules.

**Proposition 4** *The inclusion and incomparability relations among the rules we introduced are as in Table 4: if $F_1$ is the row rule and $F_2$ is the column rule, then*

- $F \subseteq F'$ *means that for every agenda $\mathcal{A}$, for every $\Gamma \in \mathcal{L}$, for every $(\mathcal{A}, \Gamma)$-profile $P$, we have $F_{\mathcal{A},\Gamma}(P) \subseteq F_{\mathcal{A},\Gamma}(P)$.*
- $F$ `inc` $F'$ *means that neither $F \subseteq F'$ nor $F' \subseteq F$*
- $F \subset F'$ *means that $F \subseteq F'$ and $F \neq F'$*

| | MCSA | MWA | RA | Y | MNAC |
|---|---|---|---|---|---|
| MSA | $\subset$ | $\subset$ | $\subset$ | inc | inc |
| MCSA | | inc | inc | inc | inc |
| MWA | | | inc | inc | inc |
| RA | | | | inc | inc |
| Y | | | | | inc |
| MNAC | | | | | |

Table 4: (Non)inclusion relationships between the rules.

The proof of this proposition can be found in Appendix.

### 3.4 A note on rules that are not majority-preserving

The Duddy-Piggins judgment aggregation rule [8] is defined as follows: the *geodesic graph* $G_{\mathcal{A}}$ associated with agenda $\mathcal{A}$ is the graph whose vertices are the consistent judgment sets over $\mathcal{A}$, and containing edge $(J, J')$ for all consistent $J, J'$ such that there is no $J$" such that $J \Delta J" \subseteq J \Delta J'$ (where $\Delta$ denotes symmetric difference). The geodesic distance $d_g$ between consistent judgment sets is defined as the length of the shortest path in $G_{\mathcal{A}}$. The distance $D(J, P^*)$ between a consistent judgment set $J^*$ and a profile $P$ is $\Sigma_{J \in P} d_g(J^*, J)$. Finally, $F_{DP}(P) = \text{argmin}_{J \in \mathcal{D}(\mathcal{A},\Gamma)} D(J, P)$. Note that the distance proposed by Duddy and Piggins [8] is also known as the geodesic metric [3, p.104].

**Proposition 5** *$F_{DP}$ is not majority-consistent.*

*Proof* Let $P$ be the profile on Table 5.

There are eight judgment sets over $\mathcal{A}$. Denote $\mathcal{D}(\mathcal{A}, \top) = \{J_1, \ldots, J_8\}$. We can verify that for every $J_i, J_j \in \mathcal{D}(\mathcal{A}, \top)$, if $J_i \neq J_j$ then the geodesic distance between $J_i$ and $J_j$ is 1. Therefore, $d_g(J_2, P) = 8$. Note that for all $J_i \in \{J_1, J_3, J_4, J_5\}$, $d_g(J_i, P) = 9$. Also, for all $J_i \in \{J_6, J_7, J_8\}$, $d_g(J_i, P) = 11$. Therefore, $F_{DP}(P) = \{J_2\}$ whereas $P$ is majority-consistent and $m(P) = \{J_1\}$.

| Voters | $\{a,$ | $b,$ | $c,$ | $a \leftrightarrow b,$ | $a \leftrightarrow c,$ | $b \leftrightarrow c\}$ |
|---|---|---|---|---|---|---|
| $2 \times J_1$ | + | + | + | + | + | + |
| $3 \times J_2$ | - | + | + | - | - | + |
| $2 \times J_3$ | + | - | + | - | + | - |
| $2 \times J_4$ | + | + | - | + | - | - |
| $2 \times J_5$ | - | - | - | + | + | + |
| $m(P)$ | + | + | + | + | + | + |

Table 5

Dietrich [4] defines a general class of *scoring rules* for judgment aggregation. Given a function $s : \mathcal{A} \times \mathcal{D}(\mathcal{A}, \top) \to \mathbb{R}$, the rule $SR_s$ is defined as

$$SR_s(P) = \operatorname*{argmax}_{J \in \mathcal{D}(\mathcal{A},\top)} \sum_{\varphi \in J} \sum_{i=1}^{n} s(J_i, \varphi)$$

Five scoring functions are defined by Dietrich [4]: reversal scoring, entailment scoring, disjoint entailment scoring, minimal entailment scoring and irreducible entailment scoring. Each scoring function gives rise to a judgment aggregation rule and we can show, using counter examples, that none of these rules is majority-preserving.

Reversal scoring $rev$ is defined as $rev(J, \varphi) = \min_{J' \in \mathcal{D}(\mathcal{A},\top),\ \varphi \notin J'} d_H(J, J')$.

**Proposition 6** *The rule $SR_{rev}$ is not majority-preserving.*

*Proof* Consider the preagenda $[\mathcal{A}] = \{p, \alpha, q\}$, where $\alpha \equiv (p \vee r) \wedge (r \vee q) \wedge (p \vee q)$ and $\Gamma = \top$. The scores according to each of the judgment sets in $\mathcal{D}(\mathcal{A}, \top)$ are given in Table 6

| | $p$ | $\neg p$ | $\alpha$ | $\neg\alpha$ | $q$ | $\neg q$ |
|---|---|---|---|---|---|---|
| $\{p, \alpha, q\}$ | 1 | 0 | 2 | 0 | 1 | 0 |
| $\{p, \alpha, \neg q\}$ | 2 | 0 | 1 | 0 | 0 | 1 |
| $\{p, \neg\alpha, \neg q\}$ | 1 | 0 | 0 | 1 | 0 | 2 |
| $\{\neg p, \alpha, q\}$ | 0 | 1 | 1 | 0 | 2 | 0 |
| $\{\neg p, \neg\alpha, q\}$ | 0 | 2 | 0 | 1 | 1 | 0 |
| $\{\neg p, \neg\alpha, \neg q\}$ | 0 | 1 | 0 | 2 | 0 | 1 |

Table 6: Reversal scores for the complete domain $\mathcal{D}(\mathcal{A}, \top)$.

| Voters | $\{p,$ | $\alpha,$ | $q\}$ |
|---|---|---|---|
| $3\times$ | + | + | + |
| $2\times$ | - | - | + |
| $1\times$ | - | + | + |
| $1\times$ | + | - | - |
| $m(P)$ | + | + | + |

Table 7: A counter example showing that reversal scoring rule is not majority-preserving.

The profile $P$ given in Table 7 is majority-consistent, with $m(P) = \{p, \alpha, q\}$. However, $SR_{rev}(P) = \{\{\neg p, \alpha, q\}\}$.

Entailment scoring $ent$, is defined as $ent(J, \varphi) = |\{S \subseteq J \mid S \text{ entails } \varphi\}|$.

**Proposition 7** *$SR_{ent}$ is not majority-preserving.*

*Proof* We consider again the same agenda as in the proof of Proposition 6. Table 8 gives the entailment scores for each judgment according to each of the judgment sets in $\mathcal{D}(\mathcal{A}, \top)$.

| | $p$ | $\neg p$ | $\alpha$ | $\neg\alpha$ | $q$ | $\neg q$ |
|---|---|---|---|---|---|---|
| $\{p, \alpha, q\}$ | 4 | 0 | 5 | 0 | 4 | 0 |
| $\{p, \alpha, \neg q\}$ | 5 | 0 | 4 | 0 | 0 | 4 |
| $\{p, \neg\alpha, \neg q\}$ | 4 | 0 | 0 | 4 | 0 | 5 |
| $\{\neg p, \alpha, q\}$ | 0 | 4 | 4 | 0 | 5 | 0 |
| $\{\neg p, \neg\alpha, q\}$ | 0 | 5 | 0 | 4 | 4 | 0 |
| $\{\neg p, \neg\alpha, \neg q\}$ | 0 | 4 | 0 | 5 | 0 | 4 |

Table 8: Entailment scores for the complete domain $\mathcal{D}(\mathcal{A}, \top)$.

| Voters | $\{p,$ | $\alpha,$ | $q\}$ |
|---|---|---|---|
| 11× | - | - | + |
| 6× | + | + | - |
| 6× | + | + | + |
| $m(P)$ | + | + | + |

Table 9: A counter example to majority-preservation of $SC_{ent}$, $SC_{dis}$, $SC_{mid}$ and $SC_{irr}$.

The entailment scoring rule is not majority-preserving. Consider the profile in Table 9. The profile is majority-consistent and $m(P) = \{p, \alpha, q\}$, however the entailment scoring rule selects the judgment set $\{\neg p, \alpha, q\}$.

The disjoint entailment scoring $dis$, minimal entailment scoring $mie$ and irreducible entailment scoring $irr$[5] are defined as follows.The function $dis(J, \varphi)$ is the number of pairwise disjoint judgment subsets of $J$ entailing $\varphi$. The function $mie(J, \varphi)$ is defined as the number of judgment subsets of $J$ which minimally entail $\varphi$. Lastly, $irr(J, \varphi)$ is defined as the number of judgment subsets of $J$ which irreducibly entail $\varphi$. More detailed explanations for each of these functions can be found in the original paper [4, p.15-17].

**Proposition 8** *$SR_{dis}$, $SR_{mie}$ and $SR_{irr}$ are not majority-preserving.*

*Proof* The same counter example suffices for all three rules. Consider the same agenda and profile as in the proof of Proposition 7. The scores according to $dis$, $mie$ and $irr$ are the same and given below.

| | $p$ | $\neg p$ | $\alpha$ | $\neg\alpha$ | $q$ | $\neg q$ |
|---|---|---|---|---|---|---|
| $\{p, \alpha, q\}$ | 1 | 0 | 2 | 0 | 1 | 0 |
| $\{p, \alpha, \neg q\}$ | 2 | 0 | 1 | 0 | 0 | 1 |
| $\{p, \neg\alpha, \neg q\}$ | 1 | 0 | 0 | 1 | 0 | 2 |
| $\{\neg p, \alpha, q\}$ | 0 | 1 | 1 | 0 | 2 | 0 |
| $\{\neg p, \neg\alpha, q\}$ | 0 | 2 | 0 | 1 | 1 | 0 |
| $\{\neg p, \neg\alpha, \neg q\}$ | 0 | 1 | 0 | 2 | 0 | 1 |

We consider the same profile as in Figure 9. The disjoint entailment, minimal entailment and irreducible entailment scoring rules all select as collective the judgment set $\{\neg p, \alpha, q\}$.

## 4 From Judgment Aggregation to Voting Rules

Since preference aggregation can be recast as a specific case of judgment aggregation using the preference agenda [5], it is natural to expect that judgment

[5] all three our notation

aggregation rules can generalise voting rules. In this section we first define what it means for a judgment aggregation rule to generalise a voting rule and then we show that several well-known voting rules are recovered as particular cases of our judgment aggregation rules.

In this section, we assume that judgment profiles contain an *odd number* of individual judgments. The reason for this assumption is that this assumption makes the connections to voting rules more natural and easier to state.

Let $C = \{x_1, \ldots, x_q\}$ be a set of alternatives. $\mathcal{L}(C)$ is the set of all strict linear orders (that is, transitive, asymmetric and connected relations) on $C$. For $\succ \in \mathcal{L}(C)$ we denote the (singleton) set containing the best element with respect to $\succ$ as $\mathtt{top}(\succeq) = \{c \in C \mid \forall c' \in C, c \succ c'\}$.

An *n-voter profile* over $C$ is a collection $V = \langle \succ_1, \ldots, \succ_n \rangle$ of strict linear orders over $C$. An *irresolute voting rule* (or *voting correspondence*) is a function $R$ mapping every $n$-voter profile for arbitrary large $n$ into a nonempty set of alternatives $R(V) \in 2^C \setminus \{\emptyset\}$. For every pair of alternatives $(x, y) \in C$ and profile $V$, let $n_V(x, y)$ be the number of votes in $V$ ranking $x$ above $y$, and let $M(V)$ be the majority graph associated with $V$, whose vertices are $C$ and containing edge $(x, y)$ iff $n_V(x, y) > \frac{n}{2}$. $x \in C$ is a *weak Condorcet winner* for $V$ if there is no ingoing edge to $x$ in $M(V)$ .

The *Top-cycle* (TC) rule maps every profile $V$ to the set of alternatives $x \in C$ such that for all $y \in C \setminus \{x\}$, there exists a path in $M(V)$ that goes from $x$ to $y$. Equivalently, $TC(P)$ is the smallest set $S$ such that for every $x \in S$ and $y \in C \setminus S$, we have $(x, y) \in M(V)$.

A *Slater order* for $V$ is a strict linear order $\succ$ over $C$ maximising the number of $(x, y)$ such that $x \succ y$ iff $(x, y) \in M(V)$. The *Slater* rule maps a profile $V$ to the set of all alternatives that are dominating in some Slater order for $M(V)$.

The *Copeland* rule maps $V$ to the set of alternatives maximising the number $n_c(x)$ of outgoing edges from $x$ in $M(V)$.

The *ranked pairs* rule [24] is defined as follows. We define first its non-neutral version: given a tie-breaking priority, that is, a strict linear order $\rho$ over $\{(x, y) \in C^2, x \neq y\}$, the strict linear order $>_\rho$ on $\{(x, y) \in C^2, x \neq y\}$ is constructed as follows: $(x, y) >_\rho (x', y')$ iff either (a) $n_V(x, y) > n_V(x', y')$ or (b) if $n_V(x, y) = n_V(x', y')$ and $\rho$ gives priority to $(x, y)$ over $(x', y')$. Then all pairs $(x, y)$ are considered in sequence according to $>_\rho$, and we build a strict linear order $\succ_\rho$ over $C$ starting with the pair on top of $>_\rho$, and iteratively adding the current pair to $\succ_\rho$ if it does not make it cyclic. The ranked pairs winner for $V$ according to $\rho$ is the unique undominated element in $\succ_\rho$. Now, $x$ is a winner of the neutral ranked pairs rule for $V$ iff it is a winner of the non-neutral ranked pairs rule for some $\rho$. (See the recent work by Brill and Fischer [2] for a discussion on neutral and non-neutral variants of ranked pairs.)

The *maximin* rule maps $V$ to the set of alternatives that maximise

$$mm(x, V) = \min_{y \in C \setminus \{x\}} n_V(x, y).$$

Let $S_Y(x, V)$ be the minimal number of votes whose removal from $V$ makes $x$ a weak Condorcet winner. If it is not possible to make $x$ a weak Condorcet winner by removing elements of $V$, we define $S_Y(x, V) = +\infty$. The *Young* (voting) rule maps $V$ to the set of alternatives that minimise $S_Y(x, V)$.

The Kemeny distance $\delta_K$ between $\succ_i \in \mathcal{L}(C)$ and $\succ_j \in \mathcal{L}(C)$ is the number of pairs $x, y \in C$ such that $x \succ_i y$ and $y \succ_j x$. The Kemeny rule is defined as

$$Kemeny(V) = \left\{ \mathtt{top}(\succ) | \succ \in \text{argmin}_{\succ \in \mathcal{L}(C)} \sum_{i=1}^{n} \delta_K(\succ, \succ_i) \right\}$$

A specific type of agenda is the *preference agenda* associated with a set of alternatives $C$. The propositions of preference agenda are of the form $xPy$ ("$x$ preferred to $y$") [6].

**Definition 10** The preference pre-agenda associated with $C = \{x_1, \ldots, x_q\}$ is $[\mathcal{A}]_C = \{x_i P x_j \mid 1 \leq i < j \leq q\}$. The corresponding agenda is the set $\mathcal{A}_C = [\mathcal{A}]_C \cup \{\neg\varphi \mid \varphi \in [\mathcal{A}]_C\}$.

When $j > i$, $x_j P x_i$ is not a proposition of $\mathcal{A}_C$, but we will write $x_j P x_i$ as a shorthand for $\neg(x_i P x_j)$.

**Definition 11** Let $C$ be a set of alternatives and $\succ \subseteq C \times C$. To $\succ$ we associate the set $J(\succ)$, defined as follows:

$$J(\succ) = \{xPy \mid x \succ y, \text{ for } x, y \in C\}.$$

Let $V = \langle \succ_1, ..., \succ_n \rangle$ be an $n$-voter profile over $C$. The judgment aggregation profile associated with $V$ is

$$P(V) = \langle J(\succ_1), \ldots, J(\succ_n) \rangle.$$

Conversely, given a set $J \subseteq \mathcal{A}_C$, the binary relation $\succ_J$ over $C$ is defined by: for all $x_i, x_j \in C$, $x_i \succ_J x_j$ if and only if $x_i P x_j \in J$.

Now we define two preference constraints: the transitivity constraint $Tr$ and the dominating alternative, or "winner", constraint $W$. Note that they both depend on $C$. However, we do not write $Tr(C)$ nor $W(C)$ when there is no danger of confusion.

**Definition 12** Let $C$ be a set of alternatives and $\mathcal{A}_C$ the associated preference agenda, with $|\mathcal{A}_C| = m = \frac{q(q+1)}{2}$. We define the transitivity $Tr$ and dominating alternative $W$ constraints:

- $Tr = \bigwedge_{i,j,k \in \{1,\ldots,m\}} \big((x_i P x_j) \wedge (x_j P x_k) \rightarrow (x_i P x_k)\big)$
- $W = \bigvee_{i \in \{1,\ldots,m\}} \bigwedge_{j \neq i} (x_i P x_j)$

Note that any complete $Tr$-consistent judgment set is also $W$-consistent, that is, $Tr$ is stronger than $W$ when applied to complete judgment sets.

**Lemma 1** *Let $\succ$ be a binary relation over $\mathcal{A}_C$.*

- *$J(\succ)$ is $Tr$-consistent if and only if $\succ$ is acyclic;*
- *$J(\succ)$ is $W$-consistent if and only if $\succ$ has at least one undominated element.*

*Proof* $J$ is $Tr$-consistent iff $\succ_J$ can be completed into a transitive order, *i.e.*, iff $\succ_J$ is acyclic; $J$ is $W$-consistent iff some $x$ can be made a winner by adding the missing propositions $xPy$, which is possible iff some $x$ is undominated in $\succ_J$.

As a consequence of Lemma 1, any $Tr$-consistent subset of $\mathcal{A}_C$ is also $W$-consistent. Note also that $\succ_J$ is a strict linear order on $C$ if and only if $J$ is a judgment set based on $\mathcal{A}_C$ and $Tr$.

For instance, let $J = \{aPb, aPc, bPc, dPb, cPe, ePb\}$; then

$$\succ_J = \{(a,b),(a,c),(b,c),(d,b),(c,e),(e,b)\}$$

$J$ is not $Tr$-consistent because $\succ_J$ contains the cycle $b \succ_J c \succ_J e \succ_J b$. However, it is $W$-consistent: $a$ and $d$ are both undominated in $\succ_J$.

For each $x \in C$ we define $W(x) = \bigwedge_{y \in C, y \neq x}(xPy)$. Note that $W$ is equivalent to $\bigvee_{x \in C} W(x)$ and that $J$ is $W(x)$-consistent iff $x$ is undominated in $\succ_J$.

Since each vote $\succ_i$ is a strict linear order, the judgment aggregation profile associated with $V$ is well-defined, i.e. every $J(\succ_i)$ is complete and $Tr$-consistent. The collective judgment will sometimes be required to be consistent with respect to $Tr$ and sometimes only to be consistent with respect to $W$. Lemma 2 is straightforward from Definition 11.

**Lemma 2** *Given a voting profile $V$, for all $x, y \in C$, $xPy$ is in $m(P(V))$ iff $(x,y) \in M(V)$.*

**Proposition 9** *A voting profile $V$ has a Condorcet winner iff $m(P(V))$ is $W$-consistent.*

*Proof* From Lemma 2, $xPy$ is in $m(P(V))$ iff $M(V)$ contains $(x,y)$. Since $n$ is odd, $m(P(V))$ contains either $x_iPx_j$ or $x_jPx_i$ for all $i \neq j$, therefore $m(P(V)) \cup \{W\} \not\models \perp$ iff there exists $x \in C$ such that $m(P(V))$ contains $\{xPy \mid y \neq x\}$, that is, by Lemma 2 again, iff $V$ has a Condorcet winner[6].

**Definition 13** Let $C$ be a set of alternatives, $\mathcal{A}_C$ the associated preference agenda and let $\Gamma \in \{Tr(C), W(C)\}$. For judgment set $J \in \mathcal{D}(\mathcal{A}_C, \Gamma)$, let

$$Win(J) = \{x \in C \mid \text{ for every } y \in C, xPy \in J\}.$$

Let $\Gamma \in \{Tr, W\}$ and $F$ be a judgment aggregation rule. The voting rule $F_{F,\Gamma}$ induced from $F$ and $\Gamma$ is defined as $x \in F_{F,\Gamma}(P(V))$ if there is a $J \in F_{F,\Gamma}(P(V))$ such that $x \in Win(J)$, or equivalently:

$$R_{F,\Gamma}(P) = \bigcup_{J \in F_\Gamma(P(V))} Win(J).$$

[6] Note that for an even $n$, $W$-consistency would be equivalent to the existence of a *weak* Condorcet winner.

Note that for any $W$-consistent, and a fortiori for any $Tr$-consistent $J$ we have $Win(J) \neq \emptyset$, therefore Definition 13 is well-founded.

Thus, for every judgment aggregation rule $F$ we have two voting rules, obtained by requiring the collective judgment set to be acyclic, *i.e.*, consistent with $Tr$, or to have a undominated element, *i.e.*, consistent with $W$.

*Example 10* Let

$$V = \langle a \succ_1 b \succ_1 c \succ_1 d, b \succ_2 c \succ_2 a \succ_1 d, d \succ_3 c \succ_3 a \succ_3 b \rangle$$

We have $P(V) = \langle J_1, J_2, J_3 \rangle$ with $J_1 = \{aPb, aPc, aPd, bPc, bPd, cPd\}$, $J_2 = \{bPa, bPc, bPd, cPa, cPd, aPd\}$ and $J_3 = \{dPa, dPb, dPc, cPa, cPb, aPb\}$; and we have $m(P(V)) = \{aPb, bPc, cPa, aPd, bPd, cPd\}$.

Let us choose $F = MSA$ and $\Gamma = Tr$. We have $F_{Tr}(P(V)) = \{J, J', J''\}$, where $J = \{aPb, aPc, aPd, bPc, bPd, cPd\}$, $J' = \{aPb, cPa, aPd, cPb, bPd, cPd\}$ and $J'' = \{bPa, cPa, aPd, bPc, bPd, cPd\}$. Now, $Win(J) = \{a\}$, $Win(J') = \{c\}$ and $Win(J'') = \{b\}$. Therefore, $F_{MSA,Tr}(P(V)) = \{a, b, c\}$.

**Proposition 10**

*1.* $R_{MSA,Tr} = TopCycle$[7]

*2.* $R_{MSA,W} = \begin{cases} \{c\} & \textit{if } V \textit{ has a Condorcet winner } c \\ C & \textit{otherwise} \end{cases}$

*Proof* We prove the first correspondence. From Lemmas 1 and 2, $J$ is a maximal $Tr$-consistent subset of $m(P)$ iff $\succ_J$ is a maximal acyclic sub-graph of $M(V)$. Let $x \in TC(V)$; then there exists an acyclic subrelation $G$ of $M(V)$ containing, for all $y \neq x$, a path from $x$ to $y$. $G$ can be completed into a maximal acyclic subrelation $G'$ of $M(V)$, and $x$ is undominated in $G'$ (because adding an edge to any $y \neq x$ would create a cycle), therefore $G'$ corresponds to a maximal $Tr$-consistent subset $J$ of $m(P(V))$, consistent with $W(x)$, which means that $x \in F_{RMSA,Tr}(V)$. Conversely, if there is a $J \in \text{MSA}_{\mathcal{A}_C,Tr}(V)$ such that $x \in Win(J)$, then there exists a maximal $Tr$-consistent subset $J'$ of $\mathcal{A}_C$ such that $J$ is a completion of $J'$ and $\succ_{J'}$ is a maximal acyclic subrelation of $M(V)$ in which $x$ does not have any incoming edge. Assume $x \notin TC(V)$; then there is an $y$ such that there is no path from $x$ to $y$ in $M(V)$. Obviously, $(x, y) \notin M(V)$, therefore, since $M(V)$ is complete, $(y, x) \in M(V)$. Adding $(y, x)$ to $\succ_J$ results in an acyclic subrelation of $M(V)$ that contains $\succ_J$, therefore $\succ_J$ is not a maximal acyclic subset of $M(V)$, contradiction.

Now we prove the second correspondence. Assume there is no Condorcet winner. Let $x \in C$. Let $S(x)$ be the subset of $m(P(V))$ defined by $\{yPz | z \neq x, yPz \in m(P(V))\}$. $S(x)$ is $W$-consistent, because it is consistent with $W(x)$. Let us now prove that $S(x)$ is a maximal $W$-consistent subset of $m(P(V))$. By means of contradiction, assume that $S(x)$ is not a maximal subset of $m(P(V))$: then there is some element of $m(P(V)) \setminus S(x)$ that can

[7] This result has been independently proven – although stated in a very different way – in [19]: point (a) of their Proposition 3.1 states that $J \in MSA(P(V))$ if $\succ_J$ is a directed Hamiltonian chain; this implies $R_{MSA,Tr} = TopCycle$.

be added to $S(x)$ without violating $W$-consistency; now, every element of $m(P(V)) \setminus S(x)$ is of the form $yPx$. Let $S' = S(x) \cup \{yPx\}$. $S'$ is not consistent with $W(x)$. Therefore, since it is $W$-consistent, it must be consistent with $W(z)$ for some $z \neq x$. This implies that there is no $tPz \in S'$, therefore, no $tPz \in S(x)$. Now, by construction of $S(x)$, this means that there is no $tPz \in m(P(V))$, which implies that $z$ is a Condorcet winner: contradiction. Thus, it must be that $S(x)$ is a maximal $W$-consistent subset of $m(P(V))$. Note also that there is a unique $W$-consistent completion $J$ of $S(x)$. Furthermore, $Win(J) = \{x\}$.

**Proposition 11**

1. $R_{MCSA,Tr} = Slater$
2. $R_{MCSA,W} = Copeland$

*Proof* For point 1, let $J \in MCSA_{Tr}(P(V))$, hence $J \in max(m(P), Tr, |.|)$ and $\succ_J$ is an acyclic subrelation of $M(V)$. Let $>$ be a linear order extending $\succ_J$. The number of edge reversals needed to obtain $>$ from $\succ_J$ is $|m(P(V)) \setminus J|$. This number is minimal iff $J$ has a maximal cardinality. Consequently, $>$ is a Slater order for $V$. Conversely, let $>$ be a Slater order for $V$ and let $J = \{xPy \mid x > y \text{ and } xPy \in m(P(V))\}$. Because $>$ is a linear order, $J$ is $Tr$-consistent. Moreover, $|m(P(V)) \setminus J|$ is the number of edge reversals needed to obtain $>$ from $M(V)$. Since $|m(P(V)) \setminus J|$ is minimal, $|J|$ is maximal and therefore $J \in MCSA_{Tr}(P(V))$. This one-to-one correspondence between Slater orders for $V$ and maxcard acyclic subgraphs of $P(V)$ allows us to conclude.

For point 2, let $J \in max(m(P), W, |.|)$. From $J \cup \{W\} \not\models \bot$ it follows that there exists a $x \in C$ such that for every $y \in C$, $yPx \notin J$. For every $y \in C$, consider $z \in C$, $z \neq x$, such that $yPz \in m(P(V))$. Adding $yPz$ to $J$ results in a judgment set which is still $W$-consistent, therefore the maximum $W$-consistent subsets of $m(P(V))$ are of the form $J_x = m(P(V)) \setminus \{yPx, y \neq x\}$ for some $x \in C$, and such a judgment set $J_x$ is a maxcard $W$-consistent subset of $m(P(V))$ iff $|\{y \mid xPy \in m(P(V))\}|$ is maximal, *i.e.*, using Lemma 2, iff $x \in Copeland(V)$.

*Example 11* Let $V$ be such that $M(V) = \{$(a,b),(a,c),(b,c),(b,d),(c,d),(d,a)$\}$, *i.e.*,
$m(P(V)) = \{aPb, aPc, bPc, bPd, cPd, dPa\}$. The only maxcard $Tr$-consistent subset of $m(P(V))$ is $J = \{aPb, aPc, bPc, bPd, cPd\}$, and $Win(J) = \{a\}$; $a$ is also the only Slater winner for $P$. Now, $m(P(V))$ has two maxcard $W$-consistent subsets: $J$ and $J' = \{aPc, bPc, bPd, cPd, dPa\}$; $Win(J') = \{b\}$; $a$ and $b$ are also the Copeland winners for $V$.

**Proposition 12**

1. $R_{RA,Tr}$ $= ranked\ pairs$.
2. $R_{RA,W}$ $= maximin$.

*Proof* The proof of point (1) is simple, due to the ty of the definitions of ranked pairs and RA, and observing that adding $xPy$ to a current $Tr$-consistent judgment set without violating $Tr$ corresponds to adding $(x, y)$ to a current acyclic graph without creating a cycle. The proof of point (2) is more interesting. The candidate $x$ is a maximin winner if it maximizes $mm(x,V)$, or equivalently, if it minimises $\max_y n_V(y,x)$. Let $\beta = \min_x \max_y n_V(y,x)$. (Note that we have $\beta > \frac{n}{2}$ when there is no Condorcet winner.) Assume that $x$ is a Maximin winner for $V$. In order to show that $x \in \textsc{ra}_{\mathcal{A}_C,W}(P(V))$, we have to construct a strict linear order $\succ = \succ_\sigma$ on $\{xPy \mid (x,y) \in C^2, x \neq y\}$, compatible with $\succsim_{P(V)}$, such that the judgment set $J_\sigma$ obtained by following $\succ_\sigma$ is such that $x \in Win(J_\sigma)$. Let $\succ_\sigma$ be as follows:

1. the first propositions of $\succ_\sigma$ are all $uPv$ such that $n_V(u,v) > \beta$, with ties broken in an arbitrary manner;
2. the propositions that follow in $\succ_\sigma$ are all $yPz$ such that $n_V(y,z) = \beta$ and $z \neq x$;
3. the following propositions are all $yPx$ such that $n_V(y,x) = \beta$;
4. the rest of $\succ_\sigma$ does not matter.

We now follow step by step the construction of $J_\sigma$. During step (1) – corresponding to considering one by one the propositions in (1) above – we consider all the propositions $uPv$ such that $n_V(u,v) > \beta$, and all are added to $S$, because the resulting judgment set is consistent with $W(x)$, and *a fortiori* with $W$ (otherwise it would be the case that for all $y$, $n_V(y,x) > \beta$, contradicting $min_x\ max_y\ n_V(y,x) = \beta$). During step (2) all propositions $yPz$ such that $n_V(y,z) = \beta$ and $z \neq x$ are considered one by one, and they are all added to $S$, because the resulting judgment set is, each time, consistent with $W(x)$ and *a fortiori* with $W$. After steps (1) and (2), due to the fact that $\beta = min_x\ max_y\ n_V(y,x)$, $S$ contains some $yPz$ for all $z \neq x$. Step (3) considers all $yPx$ such that $n_V(y,x) = \beta$, and does not add them to $S$, because this would make it inconsistent with $W$. Finally, the propositions considered in Step (4) are not of the form $yPx$. Therefore, $x \in Win(J_\sigma)$ and $x \in \textsc{ra}_{\mathcal{A}_C,W}(P(V))$.

Conversely, let $x \in \textsc{ra}_{\mathcal{A}_C,W}(P(V))$. Let $>$ be the order refining $\succsim_{P(V)}$ such that the judgment set obtained is $J$, with $x \in Win(J)$. First, all formulas $uPv$ such that $N(P,uPv) > \beta$ are added to $S$ without creating any inconsistency with $W$. Then, $>$ must consider all propositions $zPy$ such that $N(P(V),zPy) = \beta$ and $y \neq x$, and add them all to $S$; at this point, for any $y \neq x$, a proposition $zPy$ has been considered and added to $S$, otherwise there would be an $y$ such that for no $z$ it holds that $n_V(z,y) \geq \beta$, which would contradict $\beta = \min_x \max_y n_V(y,x)$. Therefore, no propositions $zPx$ will be added to $S$ (or else $W$ would be violated). Therefore, $x$ is such that $\min_x \max_y n_V(y,x) \leq \beta$, hence $\min_x \max_y n_V(y,x) = \beta$: $x$ is a maximin winner.

*Example 12* Let $n = 9$ and $V$ such that $n_V$ is as follows:

| $n_V$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | – | 6 | 2 | 4 |
| $b$ | 3 | – | 5 | 6 |
| $c$ | 7 | 4 | – | 2 |
| $d$ | 5 | 3 | 7 | – |

(3)

The weak order $\succsim_{P(V)}$ starts with $cPa$ and $dPc$ (tied), then $aPb$ and $bPd$, then $bPc$ and $dPa$, etc. Applying $\textsc{ra}_{\mathcal{A}_C,W}$ starts by adding $cPa$ and $dPc$, whatever the choice of the strict linear order $\succ_\sigma$ refining $\succsim_P$. Next, there is a choice between $aPb$ or $bPd$. If $aPb$ is considered first (that is, if $aPb \succ_\sigma bPd$), then it is added to $S$, $bPd$ is not (because it would violate $W$-consistency), and then all other propositions except $aPd$, $bPd$ and $cPd$ are added. The other choice is similar, replacing $d$ by $b$. Therefore, $\textsc{ra}_{\mathcal{A}_C,W}(P(V))$ contains the two judgment sets

$$J_1 = \{dPc, dPa, dPb, cPa, bPc, aPb\}$$

and

$$J_2 = \{bPa, bPc, bPd, dPa, dPc, cPa\}$$

with $Win(J_1) = \{d\}$ and $Win(J_2) = \{b\}$. We check that $b$ and $d$ are also the maximin winners for $V$.

For $\textsc{mwa}$, the similarity between the distance-based procedure (equivalent to the $\textsc{mwa}$ rule, as shown in Section 3) and the Kemeny rule has been exploited [10] to obtain a characterisation of the complexity of the winner determination problem in judgment aggregation under the distance-based procedure.

**Proposition 13** $R_{MWA,Tr} = Kemeny$.

*Proof* Let $V = \langle \succ_1, \ldots, \succ_n \rangle$ be an $n$-voter profile over $C$ and $J$ a judgment set based on $\mathcal{A}_C$ and $Tr$. Since $J$ is complete and $Tr$-consistent, there is a ranking $\succ$ over $C$ such that $J = J(\succ)$. Now, $W_{P(V)}(J) = \sum_{xPy \in J(\succ)} N(P(V), xPy) = \sum_{x,y \in C | x \succ y} n_V(x, y) = \frac{n \cdot q(q-1)}{2} - \sum_{i=1}^{n} \delta_K(\succ, \succ_i)$.

Let $J \in \textsc{mwa}_{\mathcal{A}_C,Tr}(P(V))$. Then $J$ is complete and $Tr$-consistent, therefore $J = J(\succ)$ for some $\succ$, and $W_{P(V)}(J) = \frac{n \cdot q(q-1)}{2} - \sum_{i=1}^{n} \delta_K(\succ, \succ_i)$. By means of contradiction, assume $\succ$ is not a Kemeny consensus for $V$; then there is a $\succ'$ such that $\sum_{i=1}^{n} \delta_K(\succ', \succ_i) < \sum_{i=1}^{n} \delta_K(\succ, \succ_i)$ therefore a complete, $Tr$-consistent $J'$ such that $W_{P(V)}(J') > W_{P(V)}(J)$, which contradicts the assumption that $J \in \textsc{mwa}_{\mathcal{A}_C,Tr}(P(V))$.

Conversely, if $\succ$ is a Kemeny consensus, then $J(\succ)$ is a $Tr$-consistent judgment set. By means of contradiction, assume $J(\succ) \notin \textsc{mwa}_{\mathcal{A}_C,Tr}(P(V))$; then there exists a complete, $Tr$-consistent $J'$ such that $W_{P(V)}(J') > W_{P(V)}(J)$; because $J'$ is complete and $Tr$-consistent, $J' = J(\succ')$ for some $\succ'$, and $W_{P(V)}(J') = \frac{n \cdot q(q-1)}{2} - \sum_{i=1}^{n} \delta_K(\succ', \succ_i)$. Therefore, we obtain that $\sum_{i=1}^{n} \delta_K(\succ', \succ_i) < \sum_{i=1}^{n} \delta_K(\succ, \succ_i)$, which contradicts the assumption that $\succ$ is a Kemeny consensus.

We have shown that $J \in \textsc{mwa}_{\mathcal{A}_C,Tr}(P(V))$ if and only if $J = J(\succ)$ for some Kemeny consensus $\succ$. Therefore, there exists $J \in \textsc{mwa}_{\mathcal{A}_C,Tr}(P(V))$ such that $Win(J) = \{x\}$ if and only if $x \in Kemeny(P)$.

The choice of the $W$ constraint leads to an unknown voting rule, for which, interestingly, the winners maximizes the sum of the Borda score and a second term. An example of winner determination for this rule, which shows that it differs from Borda, is given in [14].

**Proposition 14** $R_{Young,W} = WeakYoung.$

*Proof* Removing a minimal number of judgments from $P(V)$ so as to make it consistent is equivalent to removing a minimal number of votes from $V$ so that the majority graph contains an undominated outcome, *i.e.*, so that there exists a weak Condorcet winner.

$F_{Young,Tr}$ does not appear to be a known voting rule. It consists of the dominating candidates in maximum cardinality sub-profiles of $P(V)$ whose majoritarian aggregation is acyclic.

Table 10 summarises our results.

| | MSA | MCSA | RA | MWA | Y |
|---|---|---|---|---|---|
| $Tr$ | Top Cycle | Slater | Ranked pairs | Kemeny | |
| $W$ | | Copeland | Maximin | | Young |

Table 10: Correspondences between voting and judgment aggregation rules

Although we focus in this paper on majority-preserving rules only, such correspondences can be worked out for other rules. In particular, if $REV$ is the *reversal scoring rule* defined by Dietrich [4], then [4, Proposition 3] allows to say that $F_{REV,Tr} = Borda$.

## 5 Unanimity and Monotonicity

We are here concerned with tackling two of the general questions outlined in the Introduction: how can we lift properties from voting rules to judgment aggregation rules and how can we classify the judgment aggregation rules with respect to the properties they satisfy.

In preference aggregation, three classes of properties can be considered [26]: those that are satisfied by most common rules (such as neutrality, anonymity, Pareto-efficiency); those that are very hard to satisfy, and whose satisfaction, under mild additional condition, implies impossibility results; and finally, those that are satisfied by a significant number of rules and violated by another significant number of rules. Things are similar in judgment aggregation: weak properties such as anonymity are clearly satisfied by all our rules, while strong properties such as systematicity and independence are clearly violated by all our rules. We focus here on two properties of the third class: *unanimity* and *monotonicity*.

### 5.1 Unanimity

Unanimity is one of the most natural relational properties in social choice stating that if all agents submit the same individual information to be aggregated, then the aggregate is precisely that information. A weak unanimity property has been defined by List and Puppe [16], for resolute judgment aggregation rules, as $f(P) = J$ whenever every profile in $P$ is $J$. A stronger anonymity property, called *unanimity principle* is defined by Dietrich and List [7] in the following way for resolute rules: for every profile $\langle J_1, \ldots, J_n \rangle \in \mathcal{D}^n(\mathcal{A}, \Gamma)$ and all $\varphi \in \mathcal{A}$, if $\varphi \in J_i$ for all individuals $i$, then $\varphi \in f(J_1, \ldots, J_n)$. We lift the unanimity principle of Dietrich and List [7] to two properties of *irresolute* judgment aggregation rules which we call *weak* and *strong unanimity*, and study whether they are satisfied by our rules.

**Definition 14 (Weak and strong unanimity)**

- $F$ satisfies *weak unanimity* (WU) if for every agenda $\mathcal{A}$, for every $\Gamma \in \mathcal{L}$, for every profile $P = \langle J_1, \ldots, J_n \rangle$ based on $\mathcal{A}$ and $\Gamma$, for all $\varphi \in \mathcal{A}$, if $\varphi \in J_i$ for all $i$, then there exists a judgment set $J \in F(P)$ such that $\varphi \in J$.
- $F$ satisfies *strong unanimity* (SU) if for every agenda $\mathcal{A}$, for every $\Gamma \in \mathcal{L}$, for every profile $P = \langle J_1, \ldots, J_n \rangle$ based on $\mathcal{A}$ and $\Gamma$, for all $\varphi \in \mathcal{A}$, if $\varphi \in J_i$ for all $i$, then for all judgment sets $J \in F(P)$ we have $\varphi \in J$.

Clearly, strong unanimity implies weak unanimity.

**Proposition 15** MCSA*,* MWA *and* MNAC *do not satisfy weak (nor strong) unanimity.*

*Proof*

1. MCSA. Consider the profile $P$ from Table 19. Note that for every $J_i$ in $P$ we have $a \in J_i$ and for every $J^* \in \text{MNAC}_{\mathcal{A},\top}(P)$ we have $a \notin J^*$.
2. MWA. Consider the following example [22]. Let $[\mathcal{A}] = \{a, a \rightarrow p_1, a \rightarrow q_1, a \rightarrow (p_1 \wedge q_1), a \rightarrow p_2, a \rightarrow q_2, a \rightarrow (p_2 \wedge q_2), a \rightarrow p_3, a \rightarrow q_3, a \rightarrow (p_3 \wedge q_3), a \rightarrow p_4, a \rightarrow q_4, a \rightarrow (p_4 \wedge q_4)\}$.
   Let the profile $P$ be as given on Table 11.
   We obtain that $\text{MWA}_{\mathcal{A},\top}(P) = \{\{\neg a, a \rightarrow p_1, a \rightarrow q_1, \neg(a \rightarrow (p_1 \wedge q_1)), a \rightarrow p_2, a \rightarrow q_2, \neg(a \rightarrow (p_2 \wedge q_2)), a \rightarrow p_3, a \rightarrow q_3, \neg(a \rightarrow (p_3 \wedge q_3)), a \rightarrow p_4, a \rightarrow q_4, \neg(a \rightarrow (p_4 \wedge q_4))\}\}$. Thus, MWA does not satisfy weak (nor strong) unanimity.
3. MNAC. Again consider the agenda and profile $P$ from Table 11. We have that $\text{MNAC}_{\mathcal{A},\top}(P) = \{\{\neg a, a \rightarrow p_1, a \rightarrow q_1, \neg(a \rightarrow (p_1 \wedge q_1)), a \rightarrow p_2, a \rightarrow q_2, \neg(a \rightarrow (p_2 \wedge q_2)), a \rightarrow p_3, a \rightarrow q_3, \neg(a \rightarrow (p_3 \wedge q_3)), a \rightarrow p_4, a \rightarrow q_4, \neg(a \rightarrow (p_4 \wedge q_4))\}\}$. Hence, MNAC does not satisfy weak (nor strong) unanimity.

**Proposition 16** MSA *satisfies weak unanimity but not strong unanimity.*

| Agenda | Voters ×1 | ×1 | ×1 | $m(P)$ | $N(P,\varphi_i)$ |
|---|---|---|---|---|---|
| $a$ | + | + | + | + | 3 |
| $a \to p_1$ | + | + | - | + | 2 |
| $a \to q_1$ | + | - | + | + | 2 |
| $a \to (p_1 \wedge q_1)$ | + | - | - | - | 1 |
| $a \to p_2$ | + | + | - | + | 2 |
| $a \to q_2$ | + | - | + | + | 2 |
| $a \to (p_2 \wedge q_2)$ | + | - | - | - | 1 |
| $a \to p_3$ | + | + | - | + | 2 |
| $a \to q_3$ | + | - | + | + | 2 |
| $a \to (p_3 \wedge q_3)$ | + | - | - | - | 1 |
| $a \to p_4$ | + | + | - | + | 2 |
| $a \to q_4$ | + | - | + | + | 2 |
| $a \to (p_4 \wedge q_4)$ | + | - | - | - | 1 |

Table 11

*Proof* Let $P$ be a profile based on agenda $\mathcal{A}$ and constraint $\Gamma$, and $\varphi \in \mathcal{A}$ on which all agents give the same judgment $\varphi$. There always exists a maximal consistent sub-agenda, with respect to set inclusion, that contains $\varphi$. Consequently there exists a judgment set in $\text{MSA}_{\mathcal{A},\Gamma}(P)$ that contains $\varphi$.

As a counter-example for MSA satisfying strong unanimity, consider the profile $P$ of Table 19. MSA does not satisfy weak unanimity since there exists $J \in \text{MSA}_{\mathcal{A},\top}(P)$ such that $\neg a \in J$. Namely, $\{\neg a, \neg(a \to (b \vee c)), \neg b, \neg c, \neg(a \to (d \vee e)), \neg d, \neg e\} \in \text{MSA}_{\mathcal{A},\top}(P)$[8].

**Proposition 17** RA *and* Y *satisfy strong (and weak) unanimity.*

*Proof*

1. RA
   Let $P$ be a profile and $Y \subseteq \mathcal{A}$ be the subset of the agenda consisting of all elements on which there is unanimity among the agents. Because individual judgment sets are consistent, the conjunction of all elements of $Y$ is consistent. Now, when computing $\text{RA}_{\mathcal{A},\Gamma}(P)$, the elements of $Y$ are considered first, and whatever the order in which they are considered, they are included in the resulting judgment set because no inconsistency arises. Therefore, for all $\alpha \in Y$ and all $J \in \text{RA}_{\mathcal{A},\Gamma}(P)$, we have $\alpha \in J$.
2. Y
   Observe that if $\alpha$ is unanimously accepted by all agents in the set $N$, it is consequently unanimously selected by all consistent subsets of $N$.

## 5.2 Monotonicity

In voting theory, the standard monotonicity property states that when the position of the winning alternative for a given profile improves in some vote,

[8] $MSA$ failing to satisfy strong unanimity is also a consequence of Theorem 2.2 in [20], which can be reformulated as: $MSA$ satisfies strong unanimity if and only if $\mathcal{A}$ does not contain a minimal inconsistent subset of size 3 or more.

*ceteris paribus*, then it remains the winner. We define below a generalization of this property for (irresolute) judgment aggregation rules.

**Definition 15 (Monotonicity)**
Let $P = \langle J_1, \ldots, J_i, \ldots, J_n \rangle$ be an $\mathcal{A}, \Gamma$-profile and $\alpha \in \mathcal{A} \setminus J_i$. Let $J'_i = (J_i \setminus \{\overline{\alpha}\}) \cup \{\alpha\}$. We say that $P = \langle J_1, \ldots, J'_i, \ldots, J_n \rangle$ is an $\alpha$-*improvement* of $P$ if $J'_i$ is $\Gamma$-consistent.

A rule $F$ satisfies *monotonicity* if for all agendas $\mathcal{A}$, $\alpha \in \mathcal{A}$, $\Gamma \in \mathcal{L}$, and $\mathcal{A}, \Gamma$-profile $P$, if for all $J \in F(P)$ we have $\alpha \in J$ then for every $\alpha$-improvement $P'$ of $P$ we have $F(P) = F(P')$.

**Proposition 18** *MSA, MCSA, MWA, and RA satisfy monotonicity.*

*Proof*

1. MSA
   Let $\mathcal{A}$ be an agenda and $\Gamma \in \mathcal{L}$. Let $P$ be a profile based on $\mathcal{A}$ and $\Gamma$. If $Y \subseteq \mathcal{A}$, we use notation $P_{\downarrow Y}$ for the restriction of $P$ on $Y$. More formally, if $P = \langle J_1, \ldots, J_n \rangle$ then $P_{\downarrow Y} = \langle J_1 \cap Y, \ldots, J_n \cap Y \rangle$.
   Let $P'$ be an $\alpha$-improvement of $P$. Suppose that for every $J \in \text{MSA}_{\mathcal{A},\Gamma}(P)$ we have $\alpha \in J$ and let us prove that $\text{MSA}_{\mathcal{A},\Gamma}(P) = \text{MSA}_{\mathcal{A},\Gamma}(P')$.
   (a) Let $J \in \text{MSA}_{\mathcal{A},\Gamma}(P)$. Let $[Y] \subseteq [\mathcal{A}]$ be a maximal for set inclusion set such that $J^* = m(P_{\downarrow Y})$ is a $\Gamma$-consistent set and $J \in \mathtt{Comp}_{\mathcal{A},\Gamma}(J^*)$. Since $\alpha$ is in every $J \in \text{MSA}_{\mathcal{A},\Gamma}(P)$ then $J^* \vdash \alpha$. Let $J^{**} = m({P'}_{\downarrow Y})$. Observe that $J^{**} = J^*$. Also, $J^{**}$ is a $\Gamma$-consistent subset of $m(P')$. Furthermore, $J^{**} \vdash \alpha$. Thus, $J \in \mathtt{Comp}_{\mathcal{A},\Gamma}(J^{**})$. Hence, $J \in \text{MSA}_{\mathcal{A},\Gamma}(P')$.
   (b) Let $J' \in \text{MSA}_{\mathcal{A},\Gamma}(P')$. Let $[Y] \subseteq [\mathcal{A}]$ be maximal for set inclusion such that $J^{**} = m({P'}_{\downarrow Y})$ is a $\Gamma$-consistent set and $J' \in \mathtt{Comp}_{\mathcal{A},\Gamma}(J^{**})$. Note that $J^{**} \vdash \alpha$. (The contrary would mean that $\overline{\alpha} \in m(P)$ or that there exists $J''' \subseteq m(P)$ such that $J'''$ is $\Gamma$-consistent and $J''' \vdash \overline{\alpha}$, which is impossible.) Thus, $\alpha \in J'$. Let $J^* = m(P_{\downarrow Y})$. Then $J^* = J^{**}$. Hence, $J' \in \mathtt{Comp}_{\mathcal{A},\Gamma}(J^*)$ i.e. $J' \in \text{MSA}_{\mathcal{A},\Gamma}(P)$.

   This shows that $\text{MSA}_{\mathcal{A},\Gamma}(P) = \text{MSA}_{\mathcal{A},\Gamma}(P')$.
2. MCSA
   Let $\mathcal{A}$ be an agenda and $\Gamma \in \mathcal{L}$. Let $P$ be a profile based on $\mathcal{A}$ and $\Gamma$. Let $P'$ be an $\alpha$-improvement of $P$. Let for every $J \in \text{MCSA}_{\mathcal{A},\Gamma}(P)$, $\alpha \in J$. The case $m(P) = m(P')$ is trivial; in the rest of the proof, we study the case $m(P) \neq m(P')$.
   (a) Let us first show that for every $S \in max(m(P), \Gamma, |.|)$ we have that: $S \in max(m(P'), \Gamma, |.|)$ or $S \cup \{\alpha\} \in max(m(P'), \Gamma, |.|)$. Note first that for every $S \in max(m(P), \Gamma, |.|)$ we have $S \vdash \alpha$ (since for every $J \in \text{MCSA}_{\mathcal{A},\Gamma}(P), \alpha \in J$). We consider all the possible cases.
      i. Case $\overline{\alpha} \in m(P)$ and $\alpha \notin m(P')$ and $\overline{\alpha} \notin m(P')$. Since $S \in max(m(P), \Gamma, |.|)$ and $\overline{\alpha} \notin S$ then $S \in max(m(P) \setminus \{\alpha\}, \Gamma, |.|)$. Since $m(P') = m(P) \setminus \{\alpha\}$ this means that $S \in max(m(P'), \Gamma, |.|)$.
      ii. Case $\overline{\alpha} \in m(P)$ and $\alpha \in m(P)$. Let $|S| = k$. Note that $S \vdash \alpha$, thus $\overline{\alpha} \notin S$. Let $S' = S \cup \{\alpha\}$. $S'$ is $\Gamma$-consistent. Let us prove that

$S' \in max(m(P'), \Gamma, |.|)$. By means of contradiction, suppose that there exists $S'' \subseteq m(P')$ such that $|S''| > k + 1$.

A. Case $\alpha \in S''$. Let $S''' = S'' \setminus \{\alpha\}$. We have $|S'''| > |S|$ hence $S \notin max(m(P), \Gamma, |.|)$. Contradiction.

B. Case $\alpha \notin S''$. Since $S'' \subseteq m(P)$ and $|S''| > |S|$ we obtain that $S \notin max(m(P), \Gamma, |.|)$. Contradiction.

iii. Case $\alpha \notin m(P)$ and $\overline{\alpha} \notin m(P)$ and $\alpha \in m(P)$. Let us suppose that $S \in max(m(P), \Gamma, |.|)$. For same reasons as in the previous case, we have that $S' = S \cup \{\alpha\}$ is a maximal for cardinality $\Gamma$-consistent subset of $m(P)$.

We conclude that for every $S \in max(m(P), \Gamma, |.|)$ it holds that: $S \in max(m(P'), \Gamma, |.|)$ or $S \cup \{\alpha\} \in max(m(P'), \Gamma, |.|)$.

(b) Let us now show that $\text{MCSA}_{\mathcal{A},\Gamma}(P) \subseteq \text{MCSA}_{\mathcal{A},\Gamma}(P')$. If $J \in \text{MCSA}_{\mathcal{A},\Gamma}(P)$ then there exists $S \in max(m(P), \Gamma, |.|)$ such that $J \in \texttt{Comp}_{\mathcal{A},\Gamma}(S)$. From (a), there exists $S' \in max(m(P'), \Gamma, |.|)$ such that $S' = S$ or $S' = S \cup \{\alpha\}$. Since $S$ is a maximal subset of $m(P)$, $|\texttt{Comp}_{\mathcal{A},\Gamma}(S)| = 1$ (i.e. it $S$ is completed by adding negations of all formulae from $m(P) \setminus S$). Note also that $\texttt{Comp}_{\mathcal{A},\Gamma}(S) = \texttt{Comp}_{\mathcal{A},\Gamma}(S')$. Hence, $J \in \text{MCSA}_{\mathcal{A},\Gamma}(P')$.

(c) Let us now prove that for every $S' \in max(m(P'), \Gamma, |.|)$, $S' \setminus \{\alpha\} \in max(m(P), \Gamma, |.|)$. Let $k$ be the cardinality of sets in $max(m(P), \Gamma, |.|)$; formally let $k$ be such that for every $S \in max(m(P), \Gamma, |.|)$ $|S| = k$. As before, we proceed by case analysis.

i. Case $\overline{\alpha} \in m(P)$ and $\alpha \notin m(P')$ and $\overline{\alpha} \notin m(P')$. Let us suppose that $S' \in max(m(P'), \Gamma, |.|)$. We claim that $S' \in max(m(P), \Gamma, |.|)$. By means of contradiction, suppose the contrary. Thus $|S'| < k$. Let $S \in max(m(P), \Gamma, |.|)$. $S \vdash \alpha$ therefore $\overline{\alpha} \notin S$. This means that $S \in max(m(P'), \Gamma, |.|)$. Thus $S' \notin max(m(P'), \Gamma, |.|)$. Contradiction.

ii. Case $\overline{\alpha} \in m(P)$ and $\alpha \in m(P')$. Let $S' \in max(m(P'), \Gamma, |.|)$. From (a), we conclude that $|S'| = k + 1$ (since for every $S \in max(m(P), \Gamma, |.|)$, $|S| = k$ and $S \cup \{\alpha\} \in max(m(P'), \Gamma, |.|)$ ). Let $S'' = S' \setminus \{\alpha\}$. $S''$ is $\Gamma$-consistent, $S'' \subseteq m(P)$ and $|S''| = k$. Thus, $S'' \in max(m(P), \Gamma, |.|)$.

iii. Case $\alpha \notin m(P)$ and $\overline{\alpha} \notin m(P)$ and $\alpha \in m(P')$. Let us suppose that $S' \in max(m(P'), \Gamma, |.|)$. Note that, from (a), we conclude that $|S'| = k+1$. Let $S'' = S' \setminus \{\alpha\}$. $S'' \subseteq m(P)$, $S''$ is $\Gamma$-consistent and $|S''| = k$, thus $S'' \in max(m(P), \Gamma, |.|)$.

(d) Let us show that $\text{MCSA}_{\mathcal{A},\Gamma}(P') \subseteq \text{MCSA}_{\mathcal{A},\Gamma}(P)$. Let $J' \in \text{MCSA}_{\mathcal{A},\Gamma}(P')$ and let $S' \in max(m(P'), \Gamma, |.|)$ be a set such that $J' \in \text{MCSA}_{\mathcal{A},\Gamma}(P')$. Let $S'' = S' \setminus \{\alpha\}$. Then $S'' \in max(m(P), \Gamma, |.|)$ and $S'' \vdash \alpha$, thus $\texttt{Comp}_{\mathcal{A},\Gamma}(S') = \texttt{Comp}_{\mathcal{A},\Gamma}(S'')$. This means that $J' \in \text{MCSA}_{\mathcal{A},\Gamma}(P)$.

3. RA

Let $\alpha \in \mathcal{A}$ and suppose that for every $J \in \text{RA}_{\mathcal{A},\Gamma}(P)$, $\alpha \in J$. Let $P'$ be an $\alpha$-improvement of $P$. Then $N(P', \alpha) > N(P, \alpha)$, $N(P', \overline{\alpha}) < N(P, \overline{\alpha})$, whereas for all $\varphi \neq \alpha, \overline{\alpha}$, $N(P', \varphi) = N(P, \varphi)$. Hence, in $\succsim_{P'}$, $\alpha$ appears either at an earlier position or in the same position as in $\succsim_P$. Therefore, if

$>'$ is a linear order refining $\succsim_{P'}$, when $\alpha$ is considered, it is added to the judgment set. Hence, for every $J' \in \text{RA}_{\mathcal{A},\Gamma}(P')$, $\alpha \in J'$.

4. MWA
Let $P$ be a profile $P = (J_1, \ldots, J_k, \ldots, J_n)$ and let the $\alpha$-reinforcement of $P$ be a profile $P' = (J_1, \ldots, J_k^*, \ldots, J_n)$. For all $J \in \text{MWA}_{\mathcal{A},\Gamma}(P)$, $D(J,P) = \sum_{J_i \in P} d_H(J, J_i) = c$
We have the following assumptions:
   - $\alpha \notin J_k$,
   - $\alpha \in J_k^*$,
   - for all $\psi \in \mathcal{A}, \psi \notin \{\alpha, \overline{\alpha}\}$ it holds $\psi \in J_k$ iff $\psi \in J_k^*$,
   - $\alpha \in J$, for all $J \in \text{MWA}_{\mathcal{A},\Gamma}(P)$.

   (a) Let us first prove that $\text{MWA}_{\mathcal{A},\Gamma}(P) \subseteq \text{MWA}_{\mathcal{A},\Gamma}(P')$. Let $J \in \text{MWA}_{\mathcal{A},\Gamma}(P)$. Then $\alpha \in J$. Observe that $D(J,P') = D(J,P) - 1$ for every $J$ such that $\alpha \in J$. Thus, for every $J^* \in \text{MWA}_{\mathcal{A},\Gamma}(P)$, we have $D(J^*, P') = c - 1$. Hence, $J \in \text{MWA}_{\mathcal{A},\Gamma}(P')$.
   (b) Let us now prove that $\text{MWA}_{\mathcal{A},\Gamma}(P') \subseteq \text{MWA}_{\mathcal{A},\Gamma}(P)$. Suppose $J' \in \text{MWA}_{\mathcal{A},\Gamma}(P')$.
      i. Case $\alpha \in J'$. Then $D(J',P) = D(J',P')+1$. Since $J' \in \text{MWA}_{\mathcal{A},\Gamma}(P')$ then $D(J',P') = c - 1$. Thus, $D(J',P) = c$. Therefore, $J' \in \text{MWA}_{\mathcal{A},\Gamma}(P)$.
      ii. Case $\alpha \notin J'$. By means of contradiction, suppose $J' \notin \text{MWA}_{\mathcal{A},\Gamma}(P)$. Thus $D(J',P) > c$. Since $\alpha \notin J'$, then $D(J',P) < D(J',P')$. The two last inequalities yield $D(J',P') > c$. Contradiction with $J' \in \text{MWA}_{\mathcal{A},\Gamma}(P')$.

**Proposition 19** Y *and* MNAC *do not satisfy monotonicity.*

*Proof*

1. Y
We use a proof by counter-example. Let the pre-agenda be $[\mathcal{A}] = \{p, q, p \wedge q, r\}$ and $\Gamma = \top$. Consider the profile $P$ in Table 12. $P$ is not majority-consistent, but removing any voter who has $p$ in her judgment set suffices to restore consistency, therefore $\text{Y}_{\mathcal{A},\top}(P) = \{\{\neg p, q, \neg(p \wedge q), r\}, \{\neg p, q, \neg(p \wedge q), \neg r\}\}$. Consider the $\neg p$-reinforcement profile $P'$, Table 13. $\text{Y}_{\mathcal{A},\top}(P') = \{\{\neg p, q, \neg(p \wedge q), r\}\}$.

| Voters | $\{p,$ | $q,$ | $p \wedge q,$ | $r\}$ |
|---|---|---|---|---|
| 2× | + | + | + | + |
| 2× | + | - | - | + |
| 1× | + | - | - | + |
| 4× | - | + | - | - |
| $m(P)$ | + | + | - | + |

Table 12

| Voters | $\{p,$ | $q,$ | $p \wedge q,$ | $r\}$ |
|---|---|---|---|---|
| 2× | + | + | + | + |
| 2× | + | - | - | + |
| 1× | - | - | - | + |
| 4× | - | + | - | - |
| $m(P)$ | - | + | - | + |

Table 13

2. MNAC

Consider the pre-agenda $[\mathcal{A}] = \{p, q, p \wedge q, p \wedge r, q \wedge s\}$ and the profile $P$ given in Table 14 with $\Gamma = \top$.

| Voters | $\{p,$ | $q,$ | $p \wedge q,$ | $p \wedge r,$ | $q \wedge s\}$ |
|---|---|---|---|---|---|
| $1 \times J_1$ | + | + | + | - | - |
| $1 \times J_2$ | + | - | - | + | - |
| $1 \times J_3$ | - | + | - | - | + |
| $m(P)$ | + | + | - | - | - |

Table 14

| Voters | $\{p,$ | $q,$ | $p \wedge q,$ | $p \wedge r,$ | $q \wedge s\}$ |
|---|---|---|---|---|---|
| $1 \times J_1$ | + | + | + | - | - |
| $1 \times J_2$ | + | - | - | - | - |
| $1 \times J_3$ | - | + | - | - | + |
| $m(P')$ | + | + | - | - | - |

Table 15

There are 6 profiles $P_i$ such that $D_H(P, P_i) = 2$ (see Table 16).

$$\text{MNAC}_{\mathcal{A},\top}(P) = \left\{ \begin{array}{l} \{p, q, p \wedge q, \neg(p \wedge r), \neg(q \wedge s)\}, \\ \{p, \neg q, \neg(p \wedge q), \neg(p \wedge r), \neg(q \wedge s)\}, \\ \{\neg p, q, \neg(p \wedge q), \neg(p \wedge r), \neg(q \wedge s)\} \end{array} \right\}$$

Observe that for every $J \in \text{MNAC}_{\mathcal{A},\top}(P)$, we have $\neg(p \wedge r) \in J$.
Consider $P'$ in Table 15, which is a $\neg(p \wedge r)$-reinforcement of $P$, but $\text{MNAC}_{\mathcal{A},\top}(P') = \{\{\neg p, q, \neg(p \wedge q), \neg(p \wedge r), \neg(q \wedge s)\}\}$, since $D_H(P', P_3) = 1$.

| $Voters$ | $p$ | $q$ | $p \wedge q$ | $p \wedge r$ | $q \wedge s$ |
|---|---|---|---|---|---|
| $1\times$ | + | − | − | − | − |
| $1\times$ | + | − | − | + | − |
| $1\times$ | − | + | − | − | + |
| $m(P_1)$ | + | − | − | − | − |

| $Voters$ | $p$ | $q$ | $p \wedge q$ | $p \wedge r$ | $q \wedge s$ |
|---|---|---|---|---|---|
| $1\times$ | + | + | + | − | − |
| $1\times$ | + | + | + | + | − |
| $1\times$ | − | + | − | − | + |
| $m(P_4)$ | + | + | + | − | − |

| $Voters$ | $p$ | $q$ | $p \wedge q$ | $p \wedge r$ | $q \wedge s$ |
|---|---|---|---|---|---|
| $1\times$ | − | + | − | − | − |
| $1\times$ | + | − | − | + | − |
| $1\times$ | − | + | − | − | + |
| $m(P_2)$ | − | + | − | − | − |

| $Voters$ | $p$ | $q$ | $p \wedge q$ | $p \wedge r$ | $q \wedge s$ |
|---|---|---|---|---|---|
| $1\times$ | + | + | + | − | − |
| $1\times$ | + | − | − | + | − |
| $1\times$ | − | − | − | − | − |
| $m(P_5)$ | + | − | − | − | − |

| $Voters$ | $p$ | $q$ | $p \wedge q$ | $p \wedge r$ | $q \wedge s$ |
|---|---|---|---|---|---|
| $1\times$ | + | + | + | − | − |
| $1\times$ | − | − | − | − | − |
| $1\times$ | − | + | − | − | + |
| $m(P_3)$ | − | + | − | − | − |

| $Voters$ | $p$ | $q$ | $p \wedge q$ | $p \wedge r$ | $q \wedge s$ |
|---|---|---|---|---|---|
| $1\times$ | + | + | + | − | − |
| $1\times$ | + | − | − | + | − |
| $1\times$ | + | + | + | − | + |
| $m(P_6)$ | + | + | + | − | − |

Table 16: The profiles $P_i$, $i \in [1, 6]$ for which $D_H(P, P_i) = 2$. Note that $D_H(P', P_3) = 1$.

## 6 Summary

We have focused on the class of majority-preserving judgment aggregation rules, which is the counterpart, for judgment aggregation, of the class of Condorcet-consistent voting rules. We have reviewed several rules, related them to the existing literature, made their relationship to voting rules explicit, compared them inclusionwise, and studied them according to two major properties, namely unanimity and monotonicity. Table 17 summarises the compliance of the judgment aggregation rules we considered with these two properties.

| Property | MSA | MCSA | RA | MWA | MNAC | Y |
|---|---|---|---|---|---|---|
| Weak Unanimity | yes | no | yes | no | no | yes |
| Strong Unanimity | no | no | yes | no | no | yes |
| Monotonicity | yes | yes | yes | yes | no | no |

Table 17

The definition and study of judgment aggregation rules is only starting, and knowing that a judgment aggregation rule specializes to a well-known voting rules (sometimes, to two well-known voting rules), as our results of Section 4 tell, is a hint that the judgment aggregation rule is a natural generalization of interesting voting rules, which is a first justification for studying it; second, it gives insights about the properties it may satisfy. In particular, a challenging question is the axiomatization of judgment aggregation rules, and for this, a good start could be to start with the axiomatization (when it exists) of the voting rule(s) into which the judgment aggregation rule degenerates.

Also, a similar study for some other judgment aggregation rules would be in order; among other specific rules studied in the literature, we left out the rule $R_{d_H,max}$ studied in [13] as well as the rules studied in [4,8], none of which is majority-preserving.

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## Appendix

We now provide the proof of Proposition 4.

1. MCSA $\subset$ MSA.
   Clearly, every maxcard $\Gamma$-consistent subset of $m(P)$ is also a maximally$\Gamma$-consistent subset of $m(P)$, which implies the first direction of the proof. To show that MSA $\not\subseteq$ MCSA, consider the profile from Example 3.
2. MWA $\subset$ MSA.
   For an agenda $\mathcal{A}$, a constraint $\Gamma \in \mathcal{L}$ and a profile $P$ based on $\mathcal{A}$ and $\Gamma$, let $J \in \text{MWA}_{\mathcal{A},\Gamma}$. Let $J^* = J \cap m(P)$. $J^*$ is $\Gamma$-consistent (as a subset of a $\Gamma$-consistent set). Let us prove that $J^*$ is a maximal for set inclusion $\Gamma$-consistent subset of $(P)$. By means of contradiction, suppose there exits $J'$ such that $J^* \subset J' \subseteq m(P)$ and that $J'$ is a maximal $\Gamma$-consistent subset of $m(P)$. Note that there is a unique judgment set $J'' \subseteq \mathcal{A}$ such that $J' \subseteq J''$. Also $W(P, J'') > W(P, J)$, since $J'' \cap m(P) \supset J \cap m(P)$. Contradiction, thus $J'$ is a maximal $\Gamma$-consistent subset of $m(P)$. Consequently, $J \in \text{MSA}_{\mathcal{A},\Gamma}(P)$.
   To show that MSA $\not\subseteq$ MWA, consider Example 3.
3. RA $\subset$ MSA.
   If $J \in \text{RA}_{\mathcal{A},\Gamma}(P)$ then, by definition of RA CORRECT, $J \cap m(P)$ is a maximal $\Gamma$-consistent subset of $m(P)$. Thus, $J \in \text{MSA}_{\mathcal{A},\Gamma}(P)$. Example 3 shows that MSA $\not\subseteq$ RA.
4. MWA is incomparable with MCSA.
   To see that MCSA $\not\subseteq$ MWA, consult Example 3. As for MWA $\not\subseteq$ MCSA, consider the example from Table 19. We see that $\text{MCSA}_{\mathcal{A},\top} = \{\{\neg a, a \rightarrow (b \vee c), \neg b, \neg c, a \rightarrow (d \vee e), \neg d, \neg e\}\}$ and that $\{a, a \rightarrow (b \vee c), \neg b, c, a \rightarrow (d \vee e), \neg d, e\} \in \text{MWA}_{\mathcal{A},\top}$.

| Voters | $\{a,$ | $b,$ | $a \wedge b\}$ |
|---|---|---|---|
| 3× | + | + | + |
| 2× | + | - | - |
| 2× | - | + | - |

Table 18

| Voters | $\{a,$ | $a \rightarrow (b \vee c),$ | $b,$ | $c,$ | $a \rightarrow (d \vee e),$ | $d,$ | $e\}$ |
|---|---|---|---|---|---|---|---|
| 1× | + | + | + | - | + | + | - |
| 1× | + | + | - | + | + | - | + |
| 1× | + | - | - | - | - | - | - |

Table 19

5. RA is incomparable with MCSA.
   Consider again the example from Table 19. $\text{MCSA}_{\mathcal{A},\top}(P) = \{\ \{\neg a, a \rightarrow (b \vee c), \neg b, \neg c, a \rightarrow (d \vee e), \neg d, \neg e\}\}$ and for every $J \in \text{RA}_{\mathcal{A},\Gamma}(P)$, $a \in J$. Thus MCSA $\not\subseteq$ RA and RA $\not\subseteq$ MCSA.
6. Y is incomparable with MSA and MCSA.
   See Example 3.
7. MWA is incomparable with RA.
   See Example 3.
8. MWA is incomparable with Y.
   See Example 3.
9. RA is incomparable with Y.
   We know from Example 3 that Y $\not\subseteq$ RA. To see that RA $\not\subseteq$ Y, consider the example from Table 20.

| Voters | $\{p,$ | $q,$ | $p \wedge q,$ | $r,$ | $s,$ | $r \wedge s,$ | $t\}$ |
|---|---|---|---|---|---|---|---|
| 1× | + | + | + | - | + | - | + |
| 3× | + | + | + | - | + | - | - |
| 4× | + | + | + | + | - | - | - |
| 2× | + | - | - | + | - | - | - |
| 4× | + | - | - | + | + | + | + |
| 4× | - | + | - | + | + | + | + |

Table 20

| Voters | $\{a,$ | $a \to (b \vee c),$ | $b,$ | $c,$ | $a \to (d \vee e),$ | $d,$ | $e\}$ |
|---|---|---|---|---|---|---|---|
| 1× | + | + | + | - | + | + | - |
| 1× | + | + | - | + | + | - | + |
| 1× | + | - | - | + | - | - | + |

Table 21: Profile $Q$

The minimal number of agents to remove to make the profile majority-consistent is two. These two agents are the two agents of the fourth row (light gray shaded). We see that $\text{Y}_{\mathcal{A},\top}(P) = \{\{p, q, p \wedge q, r, s, r \wedge s, t\}\}$ and $\text{RA}_{\mathcal{A},\top}(P) = \{\{p, q, p \wedge q, r, s, r \wedge s, t\}, \{p, q, p \wedge q, r, s, r \wedge s, \neg t\}\}$. Thus, RA $\not\subseteq$ Y.

10. MNAC is incomparable with MCSA.
Example 3 shows that MCSA $\not\subseteq$ MNAC. Let us show that MNAC $\not\subseteq$ MCSA. Consider the profile $P$ from Table 19. Recall that $\text{MCSA}_{\mathcal{A},\top}(P) = \{\{\neg a, a \to (b \vee c), \neg b, \neg c, a \to (d \vee e), \neg d, \neg e\}\}$. Considering MNAC, note that there are no majority-consistent profiles at distance 1 from $P$. Let $Q$ be the profile from Table 21.
$Q$ is majority-consistent and $D_H(P, Q) = 2$. Thus, $\{a, a \to (b \vee c), \neg b, c, a \to (d \vee e), \neg d, e\} \in \text{MNAC}_{\mathcal{A},\top}(P)$. Therefore, MNAC $\not\subseteq$ MCSA.
11. MNAC is incomparable with MSA
Consider the pre-agenda $[\mathcal{A}] = \{p, q, p \wedge q, p \wedge \neg q, \alpha_1, \alpha_2, q \wedge \neg p, \alpha_3, \alpha_4\}$, where
$\alpha_1 = p \wedge \neg q \wedge \neg q$,
$\alpha_2 = p \wedge \neg q \wedge \neg q \wedge \neg q$,
$\alpha_3 = q \wedge \neg p \wedge \neg p$,
$\alpha_4 = q \wedge \neg p \wedge \neg p \wedge \neg p$.
Let $P$ be the profile from Table 22. We obtain

| Voters | $\{p,$ | $q,$ | $p \wedge q,$ | $p \wedge \neg q,$ | $\alpha_1,$ | $\alpha_2,$ | $q \wedge \neg p,$ | $\alpha_3,$ | $\alpha_4\}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 × | + | + | + | - | - | - | - | - | - |
| 1 × | + | - | - | + | + | + | - | - | - |
| 1 × | - | + | - | - | - | - | + | + | + |
| $m(P)$ | + | + | - | - | - | - | - | - | - |

Table 22

| Voters | $\{p,$ | $q,$ | $p \wedge q,$ | $p \wedge \neg q,$ | $\alpha_1,$ | $\alpha_2,$ | $q \wedge \neg p,$ | $\alpha_3,$ | $\alpha_4\}$ |
|---|---|---|---|---|---|---|---|---|---|
| $1\times$ | - | - | - | - | - | - | - | - | - |
| $1\times$ | + | - | - | + | + | + | - | - | - |
| $1\times$ | - | + | - | - | - | - | + | + | + |
| $m(P)$ | - | - | - | - | - | - | - | - | - |

Table 23

$$\text{MSA}_{\mathcal{A},\top}(P) = \{\{p, q, p \wedge q, \neg(p \wedge \neg q), \neg\alpha_1, \neg\alpha_2, \neg(q \wedge \neg p), \neg\alpha_3, \neg\alpha_4\},\\ \{p, \neg q, \neg(p \wedge q), p \wedge \neg q, \alpha_1, \alpha_2, \neg(q \wedge \neg p), \neg\alpha_3, \neg\alpha_4\},\\ \{\neg p, q, \neg(p \wedge q), \neg(p \wedge \neg q), \neg\alpha_1, \neg\alpha_2, q \wedge \neg p, \alpha_3, \alpha_4\}\}$$

To obtain $\text{MNAC}_{\mathcal{A},\top}(P)$, we need to change the first three judgments of the first voter, obtaining the profile given in Table 23. This is the minimal change, since if either the second or the third agent change either their judgment on $p$ or their judgment on $q$, they have to change additional other three judgments. We obtain $\text{MNAC}_{\mathcal{A},\top}(P) = \{\{\neg p, \neg q, \neg(p \wedge q), \neg(p \wedge \neg q), \neg\alpha_1, \neg\alpha_2, \neg(q \wedge \neg p), \neg\alpha_3, \neg\alpha_4\}\}$.
Thus, MSA `inc` MNAC.

12. MNAC is incomparable with Y, RA and MWA.
Example 3 shows that MNAC `inc` RA and MNAC `inc` Y.
Now consider the profile $P$ from Table 23 and recall that $\text{MNAC}_{\mathcal{A},\top}(P) = \{\{\neg p, \neg q, \neg(p \wedge q), \neg(p \wedge \neg q), \neg\alpha_1, \neg\alpha_2, \neg(q \wedge \neg p), \neg\alpha_3, \neg\alpha_4\}\}$.
Observe that $\text{MWA}_{\mathcal{A},\top}(P) = \{\{p, q, p\wedge q, \neg(p\wedge\neg q), \neg\alpha_1, \neg\alpha_2, \neg(q\wedge\neg p), \neg\alpha_3, \neg\alpha_4\}\}$ since for this judgment set the weight is 17, and for the remaining three other possible judgment sets the weights are: 14 for the set of the judgment sets of the second, and third agent and 16 for the judgment set $\{\neg p, \neg q, \neg(p \wedge q), \neg(p \wedge \neg q), \neg\alpha_1, \neg\alpha_2, \neg(q \wedge \neg p), \neg\alpha_3, \neg\alpha_4\}$.
Thus MNAC `inc` MWA.